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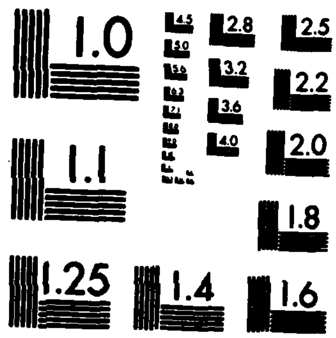


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Strong confidence intervals: A compromise between the Gaussian and the slash

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Department of Statistics
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November 15, 1983

# Strong confidence intervals: A compromise between the Gaussian and the slash

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ABSTRACT

In this report we define strong confidence interval procedures and discuss their properties. Strong confidence means that the reported confidence level is achieved even conditioned on configurations. Furthermore this is true for both the Gaussian and the slash sampling situations. We will show how such a procedure can be obtained and compare its performance to some popular non-parametric confidence intervals.

## 1. INTRODUCTION

Robust confidence interval estimation is not a heavy researched area of statistics. Around 1945 the first papers about non-parametric methods appeared. These procedures provide us with confidence intervals which reach the reported confidence coefficient under any -- or any symmetric -- sampling situation. In this sense they are robust (robustness of validity).

In this report we plan to emphasize (some more) the idea of having a highly resistant confidence coefficient. In Section 2 we

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will introduce the formulas and methods for a single, known sampling situation. Section 3, then deals with the case of unknown shape. We are concerned about the dangers of heavy-tailed situations which leads to the choice of the slash-shape as our counterpart to the Gaussian.

## 2. An equivariant confidence interval procedure for a location parameter.

A lot of work has been done in the area of robust inference about location and scale parameters. Theoretical methods like asymptotic minimax and influence curves handle the case of known scale in a convincing way and the theory is widely accepted. Hand in hand with this development went the recognition of robustness as a research topic in applied statistics. We therefore hardly have to argue in favor of robust procedures.
The need for more work connected with specific sample sizes is less widely acknowledged. The Princeton robustness study (Andrews et al(1972)) is an early example for this kind of research. There several location estimators are tested in a variety of situations and sample sizes.

In the following sections we will discuss an approach which combines the two ideas of robustness and small sample study. The topic is confidence interval estimation, where our knowledge is quite limited.

Let us start our discussion with the classic location and scale problem, where

$$\vec{y} = (y_1, y_2, y_3, \ldots, y_n)$$

is a vector of iid random variables drawn from the distribution

$$F\left(\frac{x - \mu}{\sigma}\right) \qquad \mu \in \mathbf{R},\ \sigma \in \mathbf{R}_+$$

which is fixed and known except for the location $\mu$ and the scale $\sigma$. Let us assume that the vector $\vec{y}$ is ordered, i.e.

$$y_1 \leq y_2 \leq \cdots \leq y_n \ .$$

From the beginning we restrict attention to <u>location and scale equivariant procedures</u>. By this we mean the functional property of the procedure T

$$T(s\vec{y} + r\vec{1}) = sT(\vec{y}) + r$$

whenever $s \in \mathbf{R}_+$ and $r \in \mathbf{R}$, i.e. under certain canonical transformations of the sample $\vec{y}$, the value of T transforms canonically too ($\vec{1}$ stands for the vector in $\mathbf{R}^n$ whose components are all 1). Such functions are best studied in reference to location and scale <u>configurations</u>, i.e. the equivalence classes of the equivalence relation defined on $\mathbf{R}^n$

$\vec{x}$ equivalent to $\vec{y}$ iff there is a positive, real s and a real t such that $\vec{y} = s(t\vec{1} + \vec{x})$.

Any location and scale equivariant mapping

$$T : \mathbf{R}^n \dashrightarrow \mathbf{R}$$

will be completely specified in any configuration by fixing its value

$T(\vec{c}) \in \mathbb{R}$ for a class-representing element $\vec{c}$, since by the defining relation for equivariance

$$T(\vec{y}) = T(s(t\vec{1} + \vec{c})) = sT(t\vec{1} + \vec{c}) = s(t + T(\vec{c})).$$

When doing numerical integrations (with respect to conditional distributions given the configuration) it is advisable to choose the class-representing element such that

$$c_a = -1$$

$$c_b = +1$$

and (since we have ordered from the beginning) $c_1 \leq c_2 \leq \ldots \leq c_n$. A good choice seems to be index a around $\frac{n}{4}$ and index b around $\frac{3n}{4}$. This means that we choose as our element representing the configuration, <u>that</u> sample which satisfies the two above restrictions. Of course this element will be uniquely defined.

Another way of thinking about the class-representing elements is to say that they are base points for parametrizing configurations, which are two-dimensional classes of samples.

Any sample $\vec{y} = (y_1, y_2, \ldots, y_n)$ determines uniquely its configuration (we use the word configuration also to denote the class-representing elements) by

$$c_k = \frac{y_k}{s_y} - t_y \quad ; k = 1,\ldots,n$$

where $s_y = \dfrac{y_b - y_a}{2}$ and $t_y = \dfrac{y_b + y_a}{y_b - y_a}$. And so we can parametrize $\mathbb{R}^n$ by

$$(t, s, c_1, c_2, \ldots, c_n) \quad t \in \mathbb{R}, s \in \mathbb{R}_+$$

Remember that $(c_1, c_2, \ldots, c_n) = \vec{c}$ is (n-2)- dimensional since two values are pre-fixed.

If we have a sample $\vec{y}$, we know the configuration $\vec{c}$ and it can be argued that this contains all of the "allowed" information -- we should not be influenced by the actual values of $s_y$ and $t_y$. The "point pattern" of the sample by itself should fix our ideas about location and scale. J.W. Tukey's picture for this is a code clerk who pre-processes our sample $\vec{y}$ and only gives to us the configuration $\vec{c}$ onto which we have to base our inference ("inference on the $\vec{c}$-scale"). After handing him our estimated values back, the code clerk can use his knowledge of $s_y$ and $t_y$ to transform the answer back to the $\vec{y}$-scale, he simply has to add $t_y$ back in and multiply by $s_y$.

The configurations partition the sample space $\mathbb{R}^n$ into 2-dimensional subsets -- parametrized by $t \in \mathbb{R}$ and $s \in \mathbb{R}_+$ -- which are the largest sets where a general location and scale equivariant mapping behaves perfectly simply and can be defined by a single real number.

But configurations are also connected with the probability structure on the sample space induced by iid sampling from $F(\frac{x-\mu}{\sigma})$ $\mu \in \mathbb{R}$ , $\sigma \in \mathbb{R}_+$. The idea works because of the equivariance properties of the probability structure, i.e. the canonical changes which take place inside configurations if the parameter values are changed.

Let $k_F(t, s | \mu, \sigma, \vec{c})$ denote the conditional density given the configuration $\vec{c}$. We have to find the Jacobian of the transformation

$$y_i = s(t + c_i) \quad i = 1,\ldots,n.$$

It we take the coordinates of y in the order $(y_a, y_b, y_1, \ldots, y_n)$ the Jacobian is equal to:

$$\text{abs}(\det\begin{pmatrix} s & t-1 & \cdot \\ s & t+1 & \cdot \\ s & \cdot & sI_{n-2} \end{pmatrix}) = 2s^{n-1}.$$

Hence we have

$k_F(s,t|\mu,\sigma,\vec{c})$ dsdt is proportional to

$$s^{n-1}\prod_{i=1}^{n}\{f(\frac{(s(t+c_i)-\mu)}{\sigma})\}\frac{1}{\sigma^n}\,ds\,dt,$$

i.e.proportional to $(\frac{s}{\sigma})^{n-1}\prod_{i=1}^{n}\{f(\frac{s}{\sigma}(\frac{st-\mu}{s}+c_i))\}\frac{ds}{\sigma}dt,$

where f is the density function corresponding to F.

So $k_F(s,t|\mu,\sigma,\vec{c})\,ds\,dt = k_F(p,q|0,1,\vec{c})\,dp\,dq$ if we put

$$p = \frac{s}{\sigma}$$

$$q = \frac{st-\mu}{s} = t - \frac{\mu}{s} \qquad (2.1)$$

or

$$s = p\sigma$$

$$t = q + \frac{\mu}{p\sigma}$$

We learn from this that the optimal choice of location and scale equivariant procedures is not dependent on the true parameter values $\mu$ and $\sigma$. By coding the data in terms of a configuration, we have got

rid of the "true" parameter values, since any choice we make on the $\vec{c}$-level has to be used for all possible "true" parameter values.

Figures 2.1 through 2.4 show a series of contourplots in the (t,s)-plane of the densities $k(s,t|0,1,\vec{c})$ for two configurations in twenty dimensions. Fig. 2.1 and 2.2 show the Gaussian and slash (see 3.1) in a Gaussian- drawn, nicely behaved configuration. Fig. 2.3 and 2.4 do the same with a slash-drawn configuration. The configuration is included in these pictures with "*"'s, the scale on the t-axis is, however, not relevant. We can see how the "outliers" in the second case influence the Gaussian density, which is pushed to small s-values and very stretched in the t-direction. The slash picture does not exhibit such drastic changes. Of special interest is the difference between the two densities in any given, fixed configuration, because a big such difference tells us, that the two sampling situations "interpret" the data quite differently and lead us to quite different conclusions. The configurations used for these plots are :

For the Gaussian-drawn $\vec{c}$=(-1.4,-1.3,-1.2,-1.1,-1.0,-.91, -.76,-.63,-.30,-.14,-.03,.23,.43,.64,.89,1.0,1.5,1.5,1.6, 3.2)

For the slash-drawn $\vec{c}$=(-13.8,-6.7,-1.5,-1.1,-1.0,-.90,-.65, -.51,-.39,-.09,-.01,.32,.70,.78,.91,1.0,1.2,1.3,2.8,9.5).

<u>example</u>:

If we want to find the location estimate T with minimal conditional mean square error, we have to choose $T(\vec{c})$ -- i.e. the

Figure 2.1: Contour plot of $k(s,t|0,1,\vec{c})$, the configuration is indicated by "*"'s

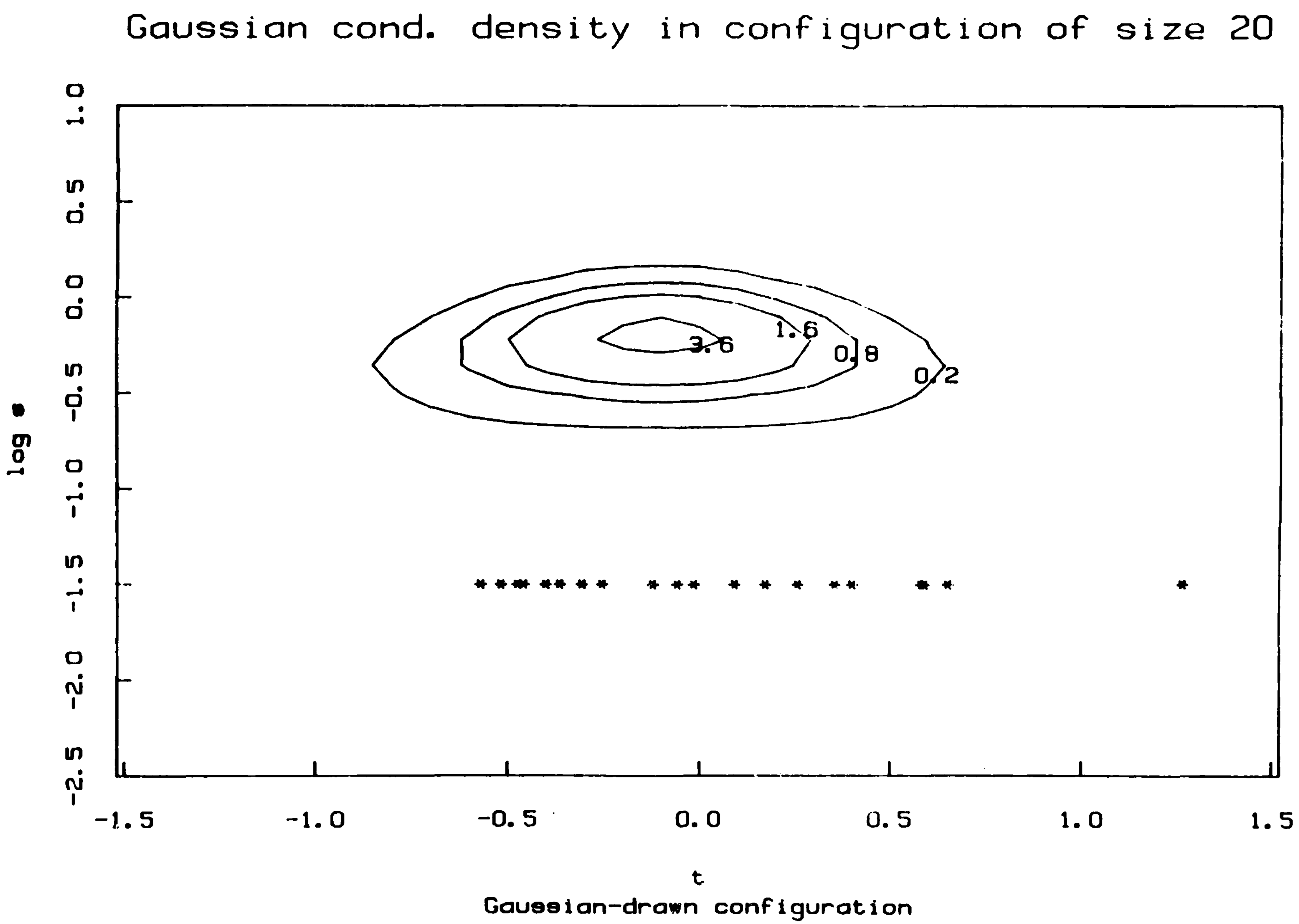


Figure 2.2: Contour plot of $k(t,s|0,1,\vec{c})$, the configuration is indicated by "*"'s

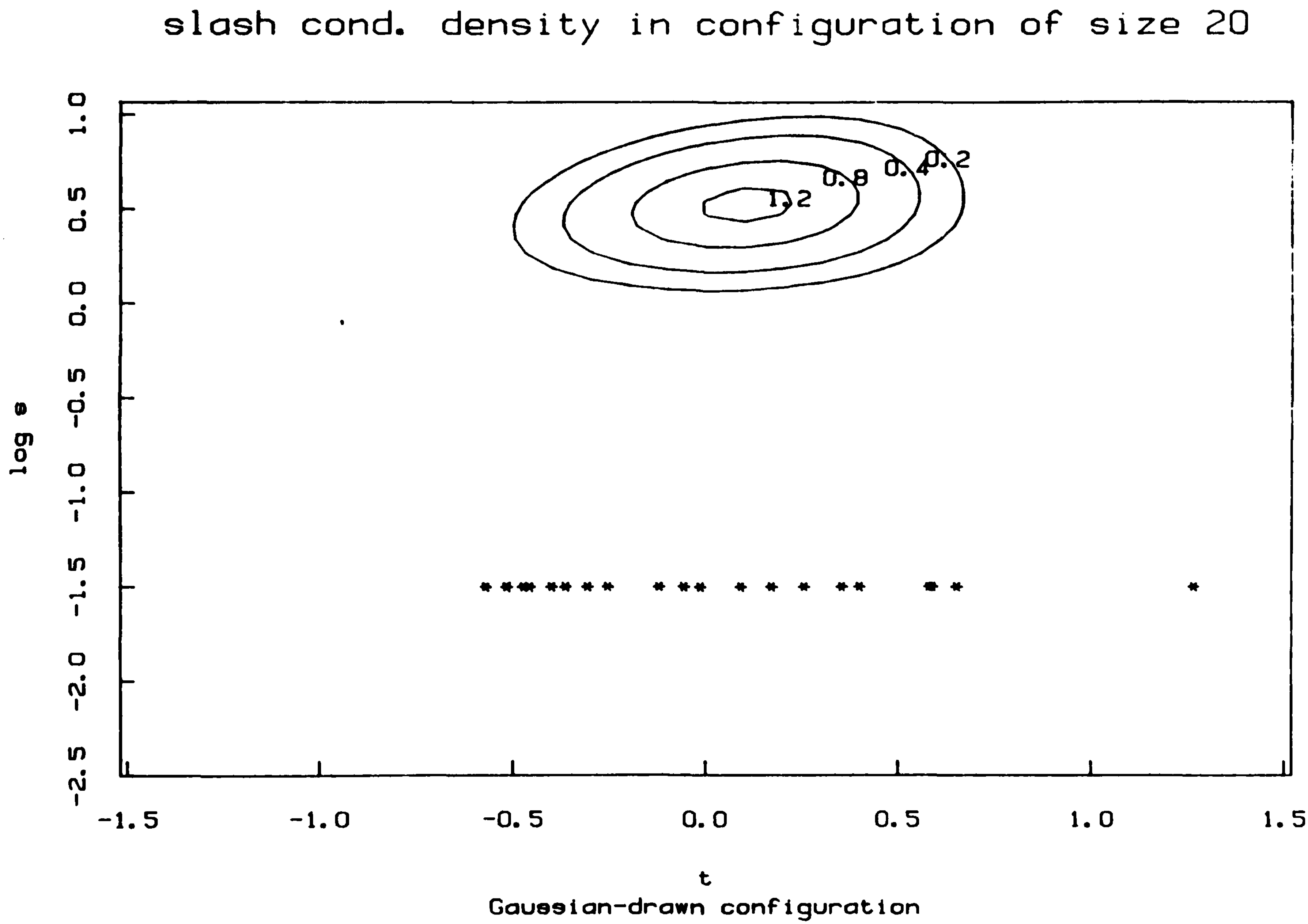


Figure 2.3: Contour plot of $k(t,s|0,1,\vec{c})$, the configuration is indicated by "*"'s

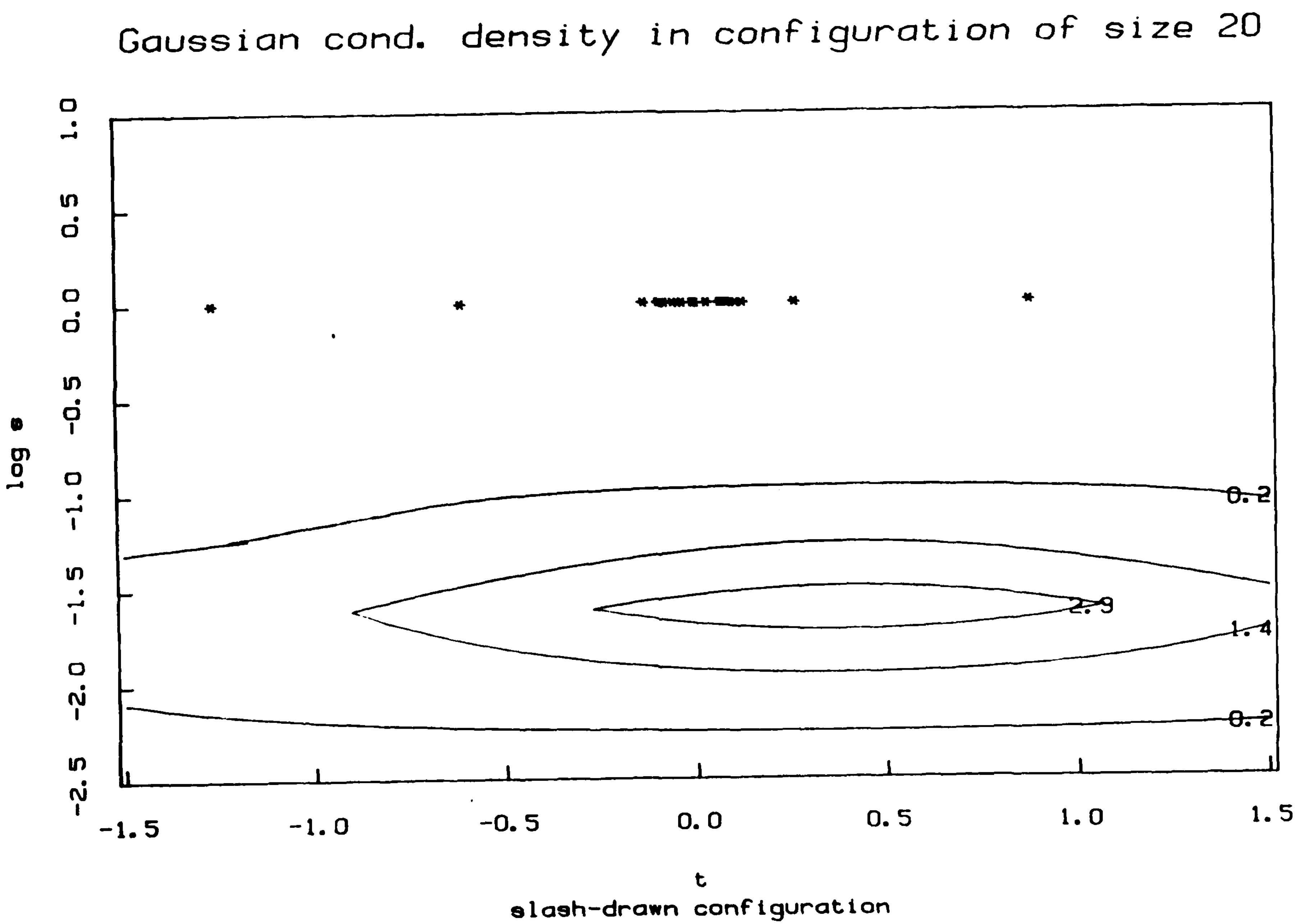
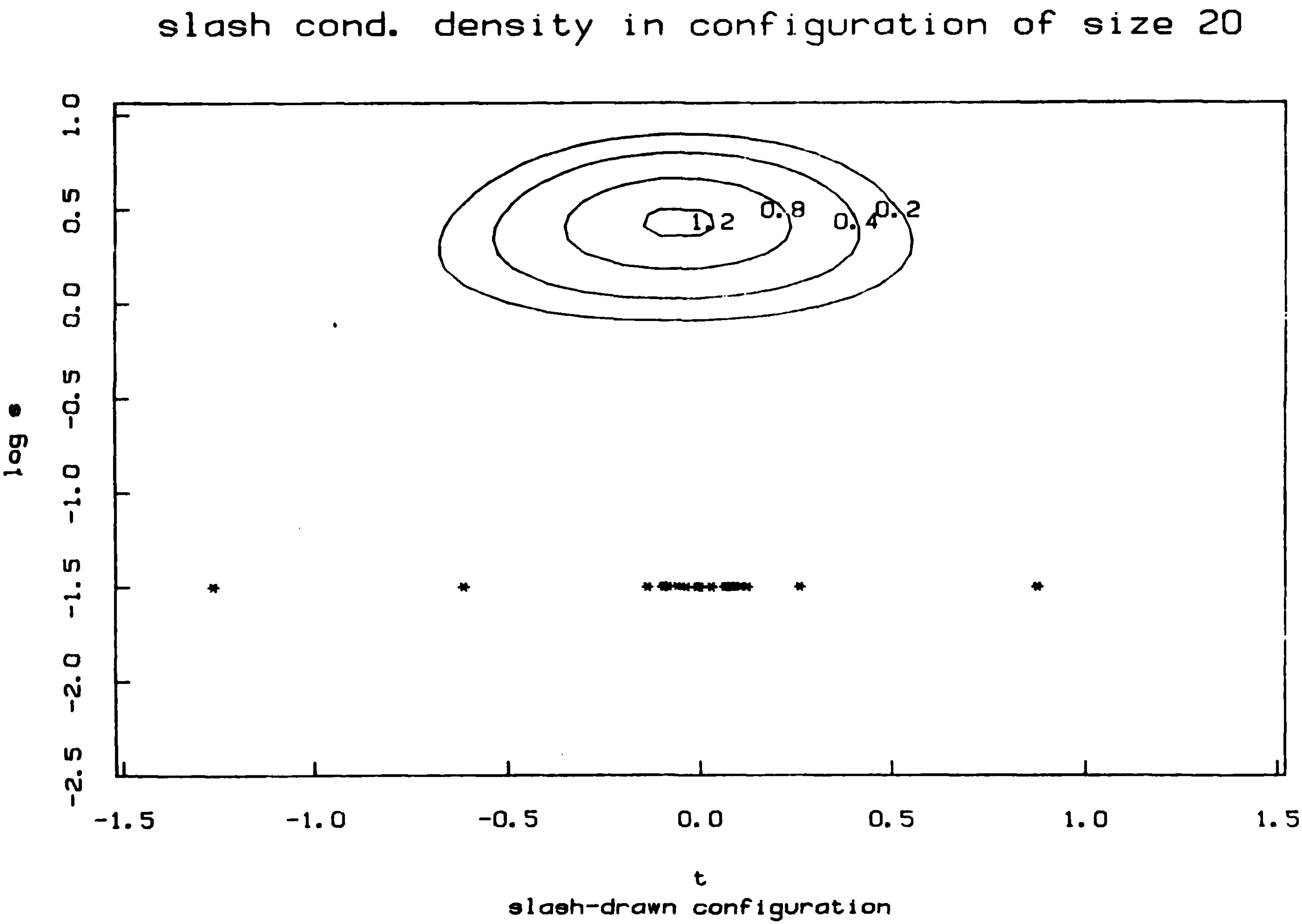


Figure 2.4: Contour plot of $k(t,s|0,1,\vec{c})$, the configuration is indicated by "*"'s



estimated value on the $\vec{c}$-scale -- such that

$$\mathrm{ave}[((T(\vec{c}) + t)s - \mu)^2 | \mu,\sigma,\vec{c}] =$$

$$\mathrm{ave}[(T(\vec{c}) + t)^2 s^2 - 2(T(\vec{c}) + t)s\mu + \mu^2 | \mu,\sigma,\vec{c}]$$

is minimal. Hence

$$\mathrm{ave}[(T(\vec{c}) + t)s^2 | \mu,\sigma,\vec{c}] = \mathrm{ave}[s\mu | \mu,\sigma,\vec{c}]$$

or

$$T_{\mu,\sigma}(\vec{c}) = \frac{\mu\ \mathrm{ave}[s|\mu,\sigma,\vec{c}] - \mathrm{ave}[ts^2|\mu,\sigma,\vec{c}]}{\mathrm{ave}[s^2|\mu,\sigma,\vec{c}]}$$

$$= \frac{\int_0^\infty \int_{-\infty}^\infty s(\mu - ts)\ k_F(s,t|\mu,\sigma,\vec{c})\ dsdt}{\int_0^\infty \int_{-\infty}^\infty s^2 k_F(s,t|\mu,\sigma,\vec{c})\ dsdt}$$

$$= \frac{-\int_0^\infty \int_{-\infty}^\infty p(qp)\, k_F(p,q|0,1,\vec{c})\ dpdq}{\int_0^\infty \int_{-\infty}^\infty p^2\ k_F(p,q|0,1,\vec{c})\ dpdq} \quad \text{(see 1.1)}$$

$$= \frac{-\mathrm{ave}[qp^2|0,1,\vec{c}]}{\mathrm{ave}[p^2|0,1,\vec{c}]} = T_{0,1}(\vec{c}). \tag{2.2}$$

This invariance result is intuitively obvious: if a code clerk gives the configuration $\vec{c}$ to the statistician and asks for a "good" location estimate, he has to come up with the same answer no matter what the true parameter values $\mu$ and $\sigma$.

For further discussion of this problem see: Pitman(1938),

Fraser(1979), Pregibon and Tukey(1981), Bell and Morgenthaler(1981).

Since the values of the parameters $\mu$ and $\sigma$ do not matter, we will always calculate our integrals with a standard form $\mu = 0$ , $\sigma = 1$ and otherwise not mention them.

The observation that the configuration $\vec{c}$ contains all of the allowed information is formalized by the statistical notion of ancillarity. The distribution across the 2-dimensional configurations, i.e. the frequencies of falling into configurations, only depends on the shape F( ) but not on the actual values of $\mu$ and $\sigma$. This is true since the step from the sample $\vec{y}$ to the configuration $\vec{c}$ involves the subtraction of a "location" estimate and division by a "scale" estimate, so that the distribution of $\vec{c}$ no longer involves the parameters $\mu$ and $\sigma$ *(and hence is pure shape)*. This fact is used as an argument to condition on the ancillary statistic $\vec{c}$ -- the conditional distribution contains the "information" about location and scale (see: Fisher(1934)). These are the ideas we want to use. They greatly simplify all inference problems about $\mu$ and $\sigma$. Firstly all our mappings will be uniquely defined on each configuration $\vec{c}$ by a single real number $t_c$. But secondly the choice of $t_c$ is often possible without reference to other configurations and we hence deal with a "simple" 2-dimensional problem no matter how big the sample size.

## 2.1. The single situation confidence distribution.

Let us fix a configuration $\vec{c}$ and a shape F ( ) in standard form, i.e. $\mu = 0$. Conditioned on $\vec{c}$ we deal with the 2-dimensional

conditional density

$$k_F(s,t|\vec{c}) = \frac{s^{n-1} \prod_{i=1}^{n} f(s(c_i+t))}{\int_0^\infty \int_{-\infty}^\infty s^{n-1} \prod_{i=1}^{n} f(s(c_i+t))\, ds dt} \qquad (2.3)$$

where $\vec{c} = (c_1,\ldots,c_n)$ and $f(x) = \frac{d}{dx}F(x)$.

<u>example</u>: <u>Gaussian</u> <u>case</u> ($F = \Phi$)

$$k_\Phi(s,t|\vec{c}) = \frac{s^{n-1}\exp(-\frac{s^2}{2}\sum_{i=1}^{n}(c_i+t)^2)}{(\frac{2\pi}{n})^{\frac{1}{2}}(n-3)(n-5)..(2 or (\frac{\pi}{2})^{\frac{1}{2}})} \left(\sum_{i=1}^{n}(c_i-\bar{c})^2\right)^{\frac{n-1}{2}}$$

The normalizing constant in this equation has to be understood as

$= (n-3)(n-5)\ldots2$ if n odd

$= (n-3)(n-5)\ldots1(\frac{\pi}{2})^{\frac{1}{2}}$ if n even.

Now we want to study the effects of setting an upper bound for the parameter $\mu$. If u is the value of the upper bound statistic U on the $\vec{c}$-scale, i.e. $U(\vec{c}) = u$, the statistic is defined on the whole configuration by

$$U(\vec{y}) = U(s(\vec{c}+t\vec{1})) = s(u+t)$$

(location and scale equivariance).

We are interested in the coverage probability defined by

$$Co_F(u) := P_F[U(y) > 0|\vec{c}]$$

It follows

$$Co_F(u) = P_F[s(u+t) > 0|\vec{c}] = P_F[t > -u|\vec{c}]$$

$$Co_F(u) = \int_0^\infty \int_{-u}^\infty k_F(s,t|\vec{c})\ dsdt. \tag{2.4}$$

$Co_F(u)$ <u>is the function which for a choice of an upper bound u on the $\vec{c}$-scale gives as its value the conditional probability of the upper bound statistic actually being an upper bound for the parameter $\mu$ on the given configuration</u>.

<u>remark</u>:

Again $Co_F(u)$ is not dependent on the values $\mu$ and $\sigma$ which are used to calculate the integrals. For a shape F( ) not in standard form we would have

$$Co_{\mu,\sigma,F}(u) = P_{\mu,\sigma,F}[U(y) > \mu] = P_{\mu,\sigma,F}[t > \frac{\mu}{s}-u] =$$

$$\int_0^\infty \int_{\frac{\mu}{s}-u}^\infty k_F(s,t|\mu,\sigma,\vec{c})\ dsdt$$

$$= \int_0^\infty \int_{-u}^\infty k_F(p,q|0,1,\vec{c})\ dpdq = Co_F(u) \qquad \text{(see 2.1)}$$

$(t = \frac{\mu}{s}-u \dashrightarrow q = \frac{\mu}{s}-u-\frac{\mu}{s} = -u)$.

From (2.4) we see that

$$Co_F(u) \dashrightarrow 1 \quad \text{as } u \dashrightarrow \infty$$

$$Co_F(u) \dashrightarrow 0 \quad \text{as } u \dashrightarrow -\infty$$

which is consistent with our intuition. $Co_F(u)$ is like a distribution function and can be used to define confidence intervals for the location parameter $\mu$.

Whenever we choose $L(\vec{c})$ and $U(\vec{c})$ such that $Co_F(U(\vec{c})) - Co_F(L(\vec{c})) = .95$ we have a 2-dimensional piece of a 95% confidence-interval procedure which has exactly 95% coverage probability conditioned on the configuration. The upper bound of the interval is below $\mu$ with probability $1 - Co(U(\vec{c}))$ and the lower bound is above with probability $Co(L(\vec{c}))$ and therefore the interval covers the parameter with conditional probability

$$1 - (1 - Co(U(\vec{c})) + Co(L(\vec{c}))) = Co(U(\vec{c})) - Co(L(\vec{c}))$$

If we choose to do so for each configuration, we will have a 95% confidence-interval procedure, since we get the "overall" coverage probability by averaging the conditional coverage probabilities, which in the above case are constant, across configurations.

The function $Co_F(\ )$ will be called a <u>coverage distribution</u> or a <u>confidence distribution</u>. It has the density

$$\frac{d}{du}Co_F(u) =: co_F(u) = \frac{d}{du}\int_0^\infty\int_{-u}^\infty k_F(s,t|\vec{c})\,dsdt$$

$$co_F(u) = \int_0^\infty k_F(s,-u|\vec{c})\,ds. \tag{2.5}$$

<u>remark</u>:

How does the mean of the confidence distribution compare with the minimum-mean-square-error location estimate? The mean m is equal to

$$m = \int_{-\infty}^{\infty} u \int_{0}^{\infty} k_F(s,-u|\vec{c})\, ds\, du =$$

$$-\int_{-\infty}^{\infty}\int_{0}^{\infty} t\; k_F(s,t|\vec{c})\, dsdt = -\mathrm{ave}(t|\vec{c}),$$

whereas the "optimal" location estimate $t_o$ is

$$t_o = -\frac{\mathrm{ave}(ts^2|\vec{c})}{\mathrm{ave}(s^2|\vec{c})} \qquad \text{(see 2.2)}.$$

The "optimal" estimate therefore takes the correlation between s and t into account.

The confidence distribution is not dependent upon the values of the parameters $\mu$ and $\sigma$ used for the calculation as we have already seen. Now we want to explore the behavior under changes of the class representing element $\vec{c}$.

Let us first look at the case where $\vec{d} = v\vec{c}$, i.e. our representing element is scaled by $v \in \mathbb{R}_+$. From (2.3) we have

$$k_F(s,t|\vec{d}) = \frac{s^{n-1}\prod_{i=1}^{n} f(s(d_i+t))}{\int_0^{\infty}\int_{-\infty}^{\infty} s^{n-1}\prod_{1}^{n} f(s(d_i+t))\, dsdt}$$

$$= \frac{s^{n-1}\prod_{i=1}^{n} f(s(vc_i+t))}{\int_0^{\infty}\int_{-\infty}^{\infty} s^{n-1}\prod_{i=1}^{n} f(s(vc_i+t))\, dsdt}$$

$$= \frac{(vs)^{n-1}\prod_{i=1}^{n} f(vs(c_i+\frac{t}{v}))}{\int_0^{\infty}\int_{-\infty}^{\infty} (vs)^{n-1}\prod_{i=1}^{n} f(vs(c_i+\frac{t}{v}))\, dsdt}$$

$$= \frac{(vs)^{n-1} \prod_{i=1}^{n} t(vs(c_i+\frac{t}{v}))}{\int_0^\infty \int_{-\infty}^\infty a^{n-1} \prod_{i=1}^{n} t(a(c_i+b))\ dadb} = k_F(vs,\tfrac{t}{v}|\vec{c})$$

We used the change ot variables:

$$a = vs \dashrightarrow da = vds \text{ and } b = \frac{t}{v} \dashrightarrow db = \frac{dt}{v} \dashrightarrow dadb = dsdt$$

The contidence density on the $\vec{d}$-scale is theretore

$$co_F(u|\vec{d}) = \int_0^\infty k_F(s,-u|\vec{d})\ ds = \int_0^\infty k_F(vs,-\tfrac{u}{v}|\vec{c})\ ds$$
$$= \int_0^\infty k_F(a,-\tfrac{u}{v}|\vec{c})\ \frac{da}{v} = \frac{1}{v}\ co_F(\tfrac{u}{v}|\vec{c}).$$

We see that the multiplier v behaves like a scale parameter!

Now let us consider the case where $\vec{d} = \vec{c}+w\vec{1}$, i.e. our representing element is translated in the (1,1,...,1) - direction by $w \in \mathbb{R}$. From (2.3) we have

$$k_F(s,t|\vec{d}) = \frac{s^{n-1} \prod_{i=1}^{n} t(s(d_i+t))}{\int_0^\infty \int_{-\infty}^\infty s^{n-1} \prod_{i=1}^{n} t(s(d_i+t))\ dsdt}$$

$$= \frac{s^{n-1} \prod_{i=1}^{n} t(s(c_i+w+t))}{\int_0^\infty \int_{-\infty}^\infty s^{n-1} \prod_{i=1}^{n} t(s(c_i+w+t))\ dsdt}$$

$$= \frac{s^{n-1} \prod_{i=1}^{n} t(s(c_i+w+t))}{\int_0^\infty \int_{-\infty}^\infty s^{n-1} \prod_{i=1}^{n} t(s(c_i+b))\ dsdt} = k_F(s,t+w|\vec{c}).$$

We used the change of variable: $b = t+w \rightarrow db = dt$.

The confidence density on the $\vec{d}$-scale is therefore

$$co_F(u|\vec{d}) = \int_0^\infty k_F(s,-u|\vec{d})ds = \int_0^\infty k_F(s,-(u+w)|\vec{c})ds = co_F(u+w|\vec{c}).$$

We see that a "translator" w behaves like a location parameter. The behavior under changes or transformations of the class-representing elements is again following our intuition:

$$co_F(u|v\vec{c}) = \frac{1}{v}\, co_F(\tfrac{u}{v}|\vec{c})$$

$$co_F(u|\vec{c}+w\vec{1}) = co_F(u+w|\vec{c}). \tag{2.6}$$

<u>example</u>: <u>Gaussian</u> <u>case</u> ($\underline{F} = \Phi$)

Using our expression for $k_\Phi(s,t|\vec{c})$ in formula (2.3) we get

$$co_\Phi(u) = \int_0^\infty \frac{s^{n-1}\exp(-\frac{s^2}{2}\sum_{i=1}^n (c_i-u)^2)}{(\frac{2\pi}{n})^{\frac{1}{2}}(n-3)(n-5)..(2or(\frac{\pi}{2})^{\frac{1}{2}})} \left(\sum_{i=1}^n (c_i-\bar{c})^2\right)^{\frac{n-1}{2}} ds$$

$$= \frac{(n-2)(n-4)\ ..(2or(\frac{\pi}{2})^{\frac{1}{2}})}{(\frac{2\pi}{n})^{\frac{1}{2}}(n-3)(n-5)\ ..(2or(\frac{\pi}{2})^{\frac{1}{2}})} \frac{\left(\sum_{i=1}^n (c_i-\bar{c})^2\right)^{\frac{n-1}{2}}}{\left(\sum_{i=1}^n (c_i-u)^2\right)^{\frac{n}{2}}}$$

$$= \frac{1}{s_{\bar{c}}}\, t_{n-1}(\frac{u-\bar{c}}{s_{\bar{c}}}) \tag{2.7}$$

where $s_{\bar{c}}^2 = \frac{1}{n(n-1)}\sum_{i=1}^n (c_i-\bar{c})^2$ and $t_{n-1}(\ )$ is the familiar t-density with (n-1) degrees of freedom in its standard form. We see that the confidence density in this case has a fixed shape across

configurations so that we could have chosen our class-representing elements in such a way that $s_{\bar{c}} = 1$ and $\bar{c} = 0$ in which case the confidence density would always be $t_{n-1}$ ( ) in standard form.

Figures 2.5 and 2.6 show the functions co( ) (see 2.5) for the two configurations we used already in Fig. 2.1 through 2.4. The red function shows the slash coverage density and the black one the Gaussian coverage density, i.e. a Student's-t density with 19 degrees of freedom. These densities are calculated by using a linear logistic fit to Co( ). It is clear how outliers in a configuration greatly influence the Gaussian coverage density. In the first -- Gaussian drawn case -- the differences are not big, the slash density is moved a bit to the left, which is reasonable if we look back at the configuration.

Once we know the conditional confidence distributions for a given sampling situation F, we can find a confidence interval estimator for any prefixed confidence coefficient $100(1-\alpha)\%$ in the following way.

In each configuration $\vec{c}$ we fix the upper and lower bounds such that the conditional coverage probability is equal to $100(1-\alpha)\%$ . This implies that the overall confidence level will indeed be $100(1-\alpha)\%$. We do, however, still have some freedom in the choice of the upper and lower bounds. The most natural choice treats the left and right tail of the conditional confidence distribution in a balanced way, i.e. we leave out $100\frac{\alpha}{2}\%$ from each tail. This now uniquely -- except possibly in pathological cases -- determines a confidence interval estimate.

Figure 2.5 : Conditional coverage densities; same configuration as in Fig. 1.1 and 1.2

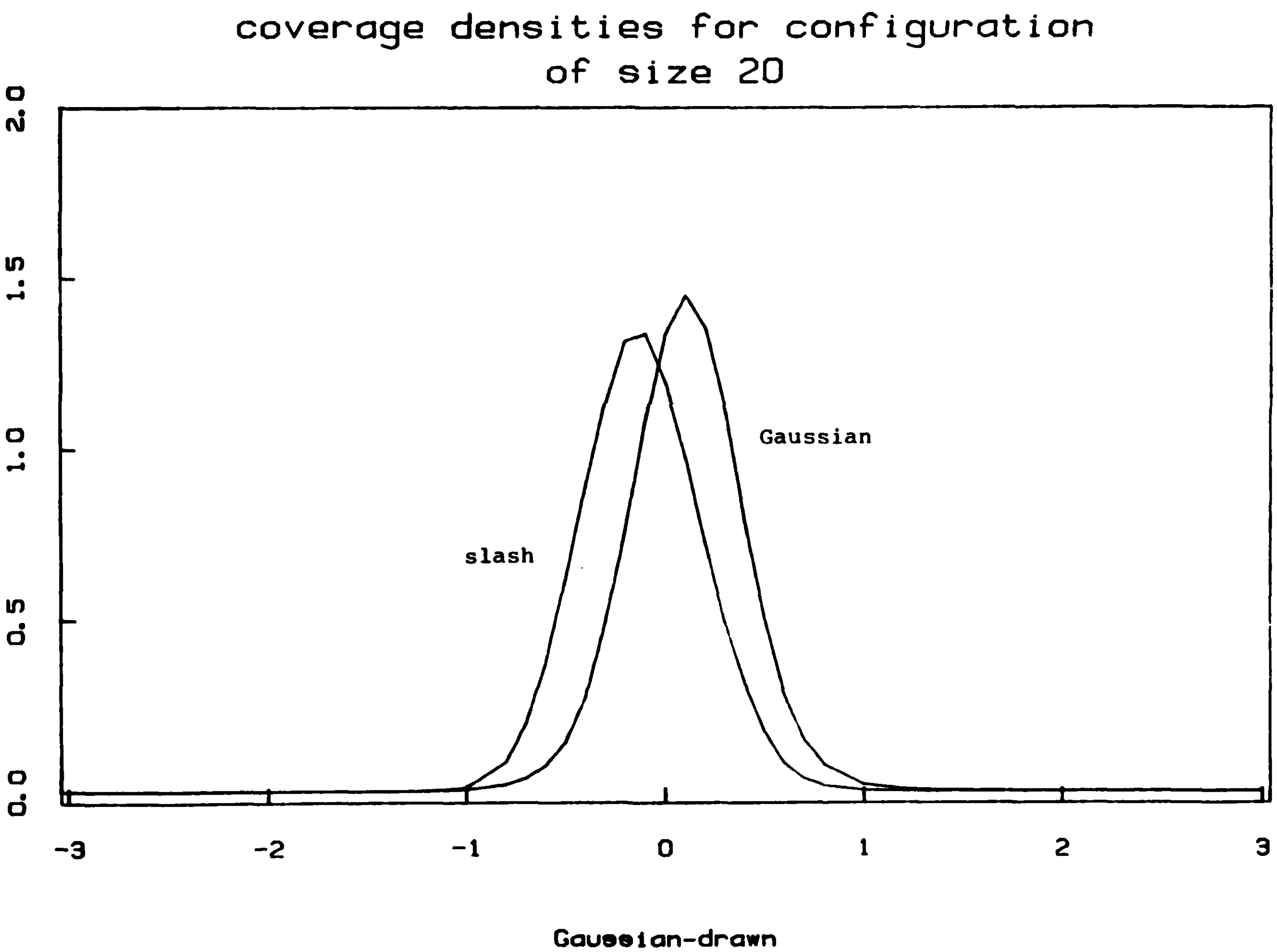
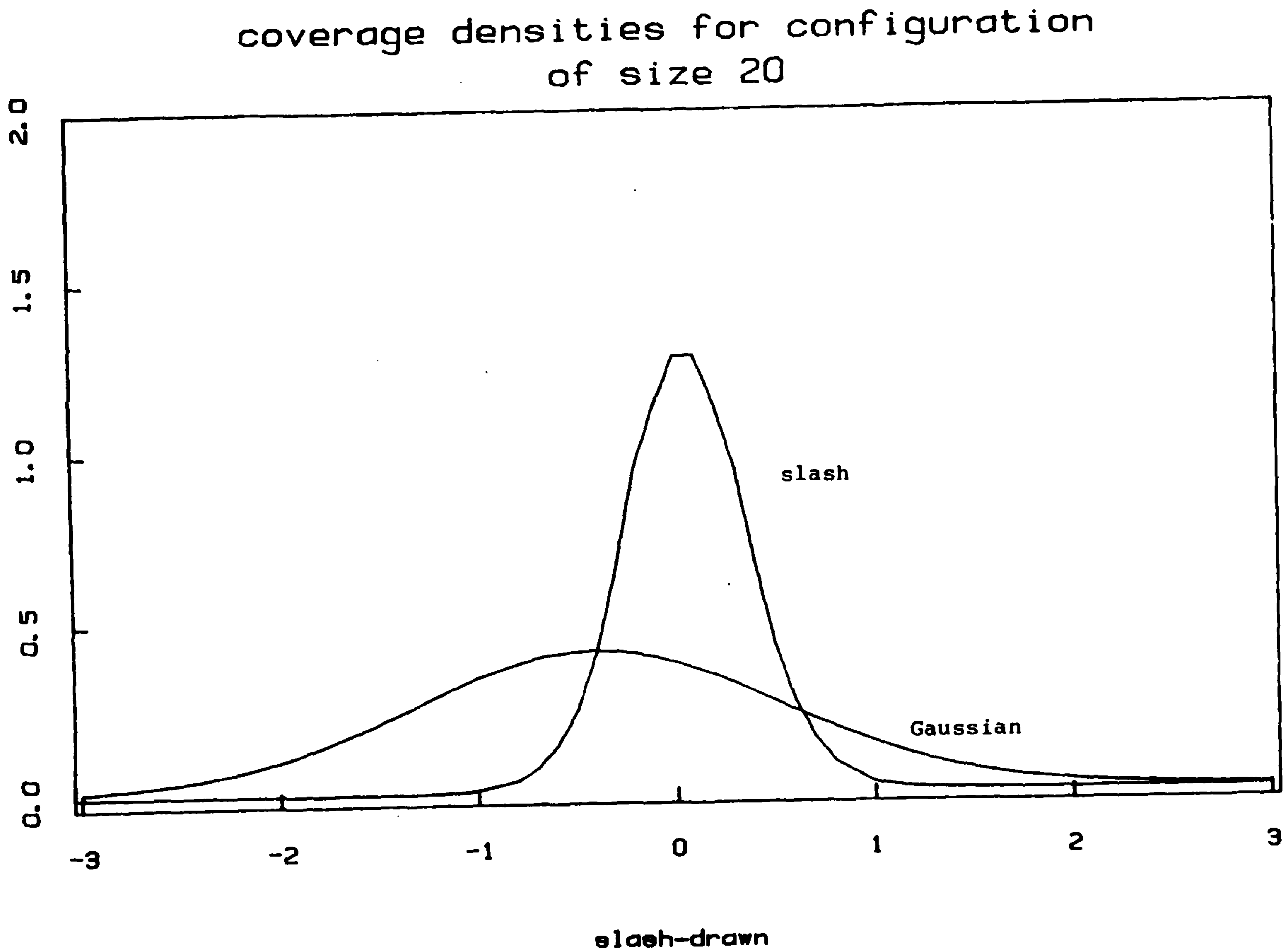


Figure 2.6: Conditional coverage densities; same configuration as in Fig. 1.3 and 1.4



Treating the two tails in a balanced way means that we care just as much about missing the true location because the lower bound is too high as missing because the upper bound is too low. Once we have fixed our idea about the allowed conditional missing probability, the balanced division seems the natural thing to do.

example: Gaussian case ($F = \Phi$)

The coverage density in this case is always a $t_{n-1}$ -density with location $\bar{c}$ and scale $s_{\bar{c}}$. The above procedure is therefore the usual symmetric t-interval.

## 3. Strong confidence intervals for a location parameter: A compromise between the Gaussian and the slash.

In this section we want to study the effects of not knowing the shape $F$ of the underlying sampling situation. In order to do this we will look at the simplest possible case where we restrict attention to two possible candidates, the Gaussian and the slash. The latter is the distribution of the ratio of a standard Gaussian and a unit uniform which are independent. The density for the slash is

$$f(x|\mu,\sigma) = \frac{\sigma}{(2\pi)^{\frac{1}{2}}(x-\mu)^2}\left[1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right] .$$

From the verbal description we recognize the slash as a "continuous" mixture of Gaussians with scales which are like an inverse uniform. The density shows us the tail behavior as $\frac{1}{x^2}$ and is therefore like a t-density with one degree of freedom ( see Rogers and Tukey (1972))

for further insights).

Nonparametric interval procedures -- like the sign interval, the Wilcoxon signed rank interval and so on -- are an attractive choice of compromise for many statisticians. These intervals are guaranteed to reach the desired confidence for all symmetric situations and hence seem to solve the problem of compromising between situations once and for all. But we would expect that seeking to put such a vast class of situations under one hat has its disadvantages. Furthermore it is not at all clear how these procedures behave conditionally (on configurations). All nonparametric tests -- from which the corresponding intervals are derived -- need an argument of equal probability under permutations. They condition on the class of samples which one gets from the one at hand under permuting around the hypothesized parameter value. A bit of thought shows us that this is an operation which does not preserve the configuration. It should therefore be revealing to learn more about the properties of nonparametric intervals conditioned on a given configuration.

It has been pointed out in the "robustness literature" that the stability of the confidence level of nonparametric procedures is only one aspect which the statistician tries to keep under control -- this property has been named "robustness of validity". Another aspect of interest is the efficiency -- which can be expressed in various ways -- of a statistical procedure.

## 3.1. Strong confidence intervals

Even if robustness of validity is our goal, we need not stop at

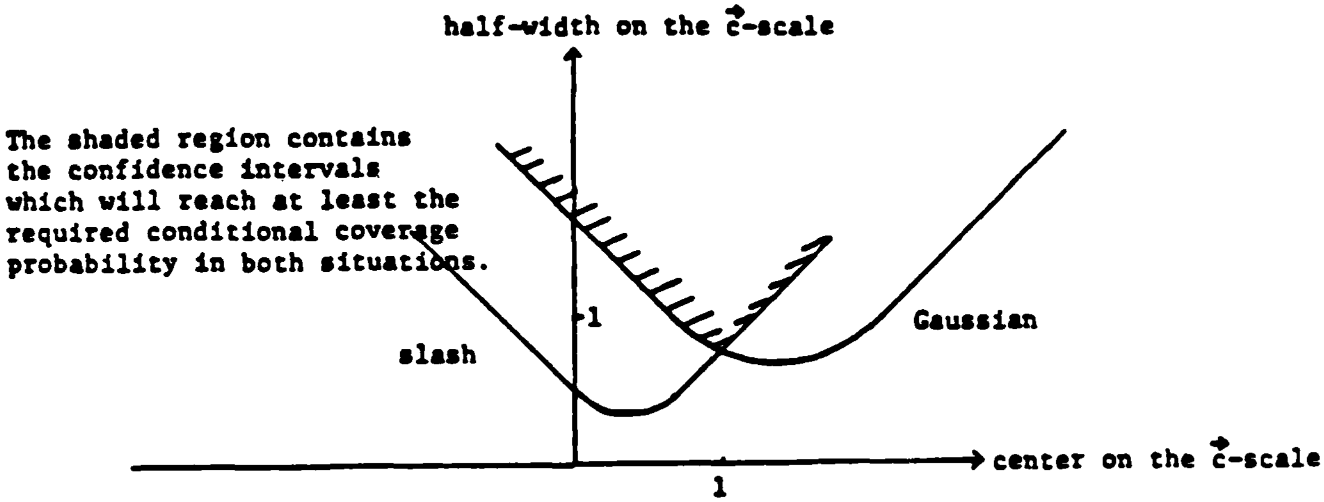
nonparametric intervals.

> We might reasonably ask for a confidence interval which has a "robust" confidence coefficient (or coverage probability) conditioned on configurations. In our "two-situation world" this would mean that <u>the intervals have to be big enough to reach 100(1-$\alpha$)% coverage probability conditioned on each configuration for the slash and for the Gaussian</u>. Of course there will usually not be such an interval and we have to settle for at least 100(1-$\alpha$)% coverage probability conditioned on each configuration for the slash and the Gaussian.

We already know that over the sample space the slash coverage of Student's t interval is conservative. The above approach might then not be very far from the classical t-interval. We can also say that the solution to this problem will end up enlarging the t-intervals in certain configurations.

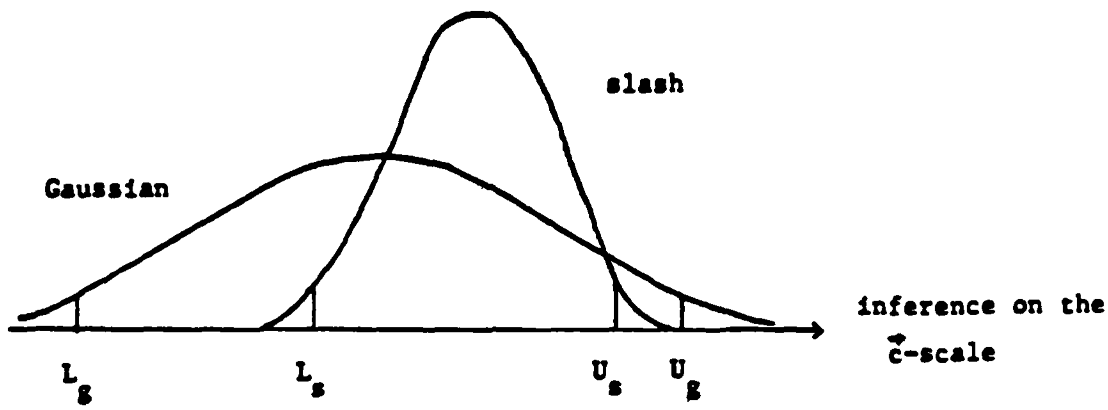
If we view robustness as a problem in stability of coverage probabilities, i.e. a problem in how safe is the use of a confidence interval, we are lead to search for rather long intervals and our flexibility in choosing upper and lower bounds is quite restricted as Figure 3.1 tries to show.

Figure 3.1: Solutionspace $[L(\vec{c}),U(\vec{c})]$ conditioned on configuration $\vec{c}=(c_1,\ldots,c_n)$




In order to study these strong interval procedures we adopt the following strategy. Our starting point for each configuration consists of the two intervals which have $100(1-\alpha)\%$ coverage probability and are treating the upper and lower tail of the coverage density symmetrically. Figure 3.2 shows the conditional situation.

Figure 3.2: Conditional coverage densities for the slash and Gaussian situations and the two symmetric intervals




In order to find a "strong" interval, i.e. one which has at least $100(1-\alpha)\%$ conditional coverage probability in both situations we will look at the interval which we get by selecting the maximum of the two upper bounds and the minimum of the two lower bounds.

$$L = \min(L_g, L_s)$$

$$U = \max(U_g, U_s) \tag{3.1}$$

(see Figure 3.2)

Clearly this will be a strong interval which can be found relatively easily. There are four possible cases into which configurations can fall:

(a) $[L,U] = [L_g,U_g]$

(b) $[L,U] = [L_s,U_s]$

(c) $[L,U] = [L_g,U_s]$

(d) $[L,U] = [L_s, U_g]$ (3.2)

The cases (a) and (b) are such that one of the two situations dominates the other, and the choice of [L,U] is more or less the best thing one can do. In the mixed cases (c) and (d) however the interval [L,U] could be shortened further and still be kept strong. The possible gains from such a shortening however are small compared to the difficulty in finding them.

We will therefore examine the intervals given by (3.1) for the case of 95% coverage probability. Table 3.1 shows the percentage of observed cases (a), (b) and (c)&(d) in the different sample sizes.

Table 3.1: Percentage of observed cases (a), (b) and (c)&(d) (for definition see (3.2))

| | Gaussian situation | | | slash situation | | |
|---|---|---|---|---|---|---|
| sample size | (a) | (b) | (c)&(d) | (a) | (b) | (c)&(d) |
| 20 | 18% | 12% | 70% | 80% | 0% | 20% |
| 10 | $32\frac{2}{3}$% | $16\frac{2}{3}$% | $50\frac{2}{3}$% | 78% | $1\frac{1}{3}$% | $20\frac{2}{3}$% |
| 5 | $89\frac{1}{5}$% | $3\frac{1}{5}$% | $7\frac{3}{5}$% | $94\frac{4}{5}$% | 1% | $4\frac{1}{5}$% |

(In samples of sizes 10 and 20 the numbers are based on 150 slash-drawn and 150 Gaussian-drawn configurations, in samples of size 5 the corresponding numbers are 500 and 500!)

The message from this table is striking. The <u>Gaussian situation is dominating in samples of size 5</u>, where we are most of the time in a case as shown in Figure 3.2 and where Student's t interval is close

to the "best" (in terms of expected length) strong interval procedure. In samples of size 5 then, we do not expect that compromising between Gaussian and heavy-tailed slash is very difficult. The slash intervals are overly optimistic and too short if judged from the Gaussian point of view. The Gaussian intervals, i.e. Student's t intervals, on the other hand seem to do a good job of keeping the conditional coverage probability above 95% even in the slash situation. And this is true whether the samples are drawn from a Gaussian or from a heavy-tailed shape. One might conclude that maybe more serious challenges to the Gaussian model for location confidence intervals have to come from less heavy-tailed shapes than the slash.

Samples of size 10 and 20 behave roughly similarly, but very differently from samples of size 5. In slash drawn samples the Gaussian situation mostly dominates, but not as overwhelmingly as in samples of size 5. In Gaussian drawn configurations the mixed cases are a majority, introducing the slash along with the Gaussian and thus really contributes a new point of view. In a lot of configurations it forces us to acknowledge the fact that Student's t interval doesn't stretch far enough to the right or to the left and has to be enlarged. A more detailed account of the behavior of the "strong" intervals from a configural point of view gives the following results.

For samples of size 20, Figure 3.3 shows the conditional coverage probabilities for the Gaussian and slash situation. Figure 3.4 is the corresponding picture for sample size 10. We see how the

Figure 3.3: Stem-and-leaf plots of the cond. confidence levels of the strong interval (3.1) for 150 sampled configurations of size 20

```
N = 150   Median = 0.96295   Hinges = 0.955, 0.9676

 Decimal point is 3 place(s) to the left of the colon

 24   24   952 : 667777788888888999999999
 35   11   953 : 00001112278
 37    2   954 : 44
 40    3   955 : 044
 48    8   956 : 22334578
 49    1   957 : 8
 52    3   958 : 777
 56    4   959 : 5889
 64    8   960 : 23455689
 69    5   961 : 11279
 75    6   962 : 124699
 75   12   963 : 025666788899
 63   11   964 : 00234447899
 52    6   965 : 113469
 46    6   966 : 005579
 40    9   967 : 056669999
 31    5   968 : 23578
 26    4   969 : 1358
 22    4   970 : 0067
 18    6   971 : 244799
 12    6   972 : 002366
  6    1   973 : 7
  5    2   974 : 29
  3    2   975 : 07
  1    0   976 :
  1    0   977 :
  1    1   978 : 8
```

Gaussian situation

```
N = 150   Median = 0.9983   Hinges = 0.9809, 1

 Decimal point is 2 place(s) to the left of the colon

  1    1    95 : 1
  2    1    95 : 7
  3    1    96 : 2
  9    6    96 : 667788
 20   11    97 : 11233444444
 37   17    97 : 55555555555567789
 43    6    98 : 122444
 48    5    98 : 55999
 57    9    99 : 002224444
      27    99 : 555556666667777888888999999
 66   66   100 : 00000000000000000000000000000000000000000000000000
```

slash situation

Figure 3.4: Stem-and-leaf plots of the cond. confidence levels of the strong interval (3.1) for 150 sampled configurations of size 10

```
N = 150   Median = 0.9559   Hinges = 0.95, 0.962

    Decimal point is 3 place(s) to the left of the colon

    52   52   950 : 0000000000000000000000000000000000000000000000000124
    56    4   951 : 6688
    62    6   952 : 135678
    67    5   953 : 11246
    73    6   954 : 344889
    75    2   955 : 08
    75    9   956 : 012237889
    66    7   957 : 0334588
    59    5   958 : 02569
    54    4   959 : 4557
    50    7   960 : 0133469
    43    5   961 : 02568
    38    9   962 : 013367789
    29    9   963 : 013555689
    20    3   964 : 579
    17    2   965 : 29
    15    4   966 : 0267
    11    2   967 : 15
     9    1   968 : 2
     8    0   969 :
     8    1   970 : 8
     7    2   971 : 23
     5    0   972 :
     5    4   973 : 0155
     1    1   974 : 9
```

Gaussian situation

```
N = 150   Median = 0.993   Hinges = 0.9737, 0.9999

    Decimal point is 2 place(s) to the left of the colon

     4    4    95 : 0[illegible]23
     8    4    95 : 6899
    16    8    96 : 00012234
    31   15    96 : 555677888888899
    44   13    97 : 0001124444444
    54   10    97 : 5556779999
    58    4    98 : 0034
    70   12    98 : 555566799999
         10    99 : 0012333344
    70   24    99 : 556677777788888888889999
    46   46   100 : 0000000000000000000000000000000000000000000000
```

slash situation

smaller sample size moves us nearer to Student's t interval, which would have exactly constant 95% conditional coverage probability in the Gaussian situation. In the slash situation this shows in a trend to be increasingly "overlong" in samples of size 20.

The following table gives the expected length of these intervals.

Table 3.2: Expected length of "strong" intervals

| | Gaussian situation | slash situation |
|---|---|---|
| size=5 | 2.335 (.0003) | 103.0 (75.6) |
| size=10 | 1.458 (.0058) | 22.12 (7.54) |
| size=20 | 1.021 (.0053) | 63.93 (52.2) |

In comparison to Table 3.1 we can again see what we noticed by looking at conditional coverage probabilities. As the sample size increases one has to enlarge Student's t interval in order to have at least 95% slash conditional coverage.

The next three pictures, Figures 3.5, 3.6 and 3.7, show us something about the conditional unbalance or asymmetry of the strong intervals. Here we plot the conditional probability of missing the true parametervalue at the low (i.e. left) end vs. at the high end.

Two things are quite noticable. In all three cases most of the strong intervals are slightly asymmetric. There is a branch from (0.0, 0.0) to (0.025, 0.025) above and below the diagonal -- which contains the balanced intervals. The correction from

Figure 3.5: Plot of the lower vs. the upper conditional missing probabilities of the strong intervals (4.1) in the slash situation

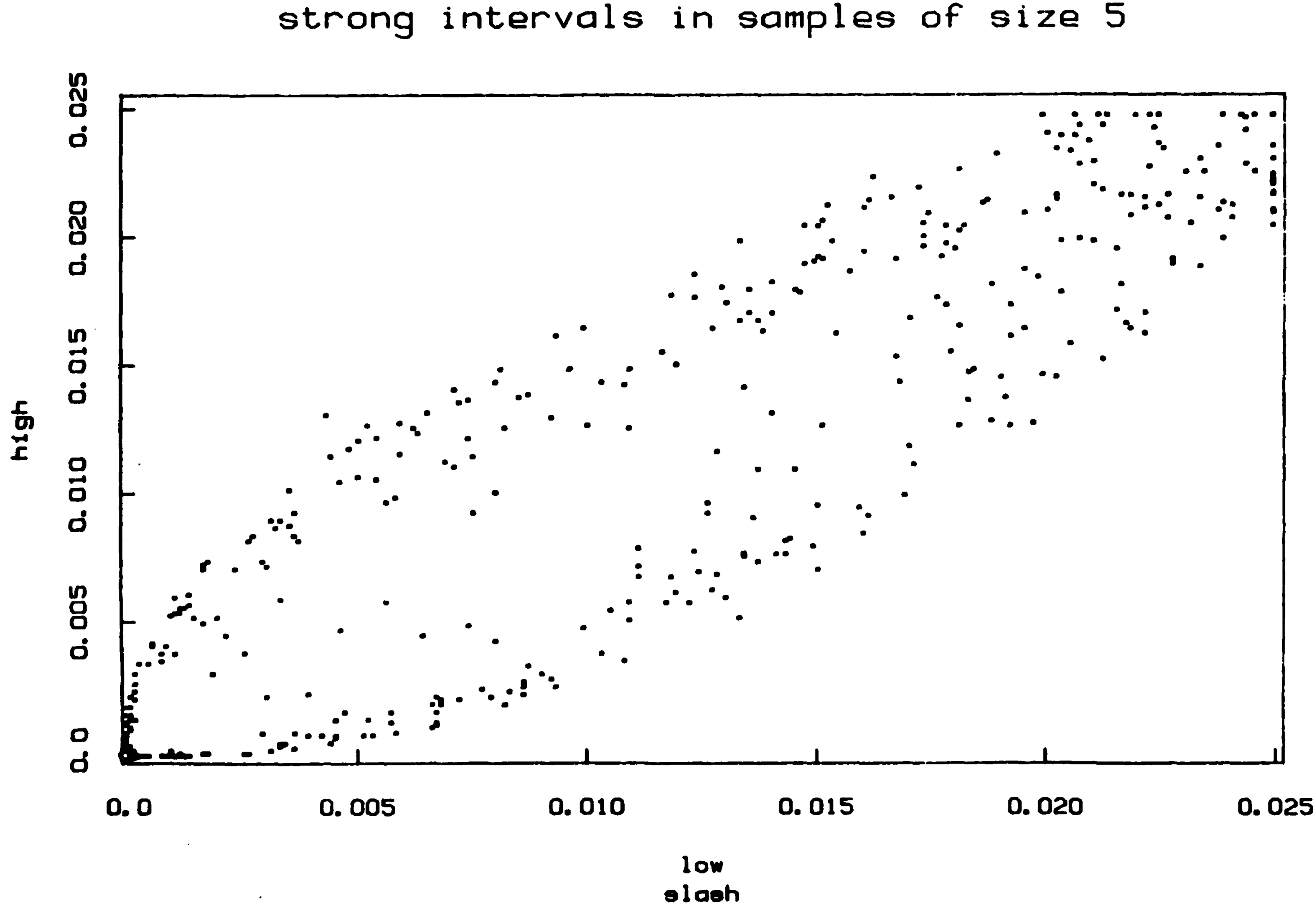


Figure 3.6: Plot of the lower vs. the upper conditional missing probabilities of the strong intervals (4.1) in the slash situation

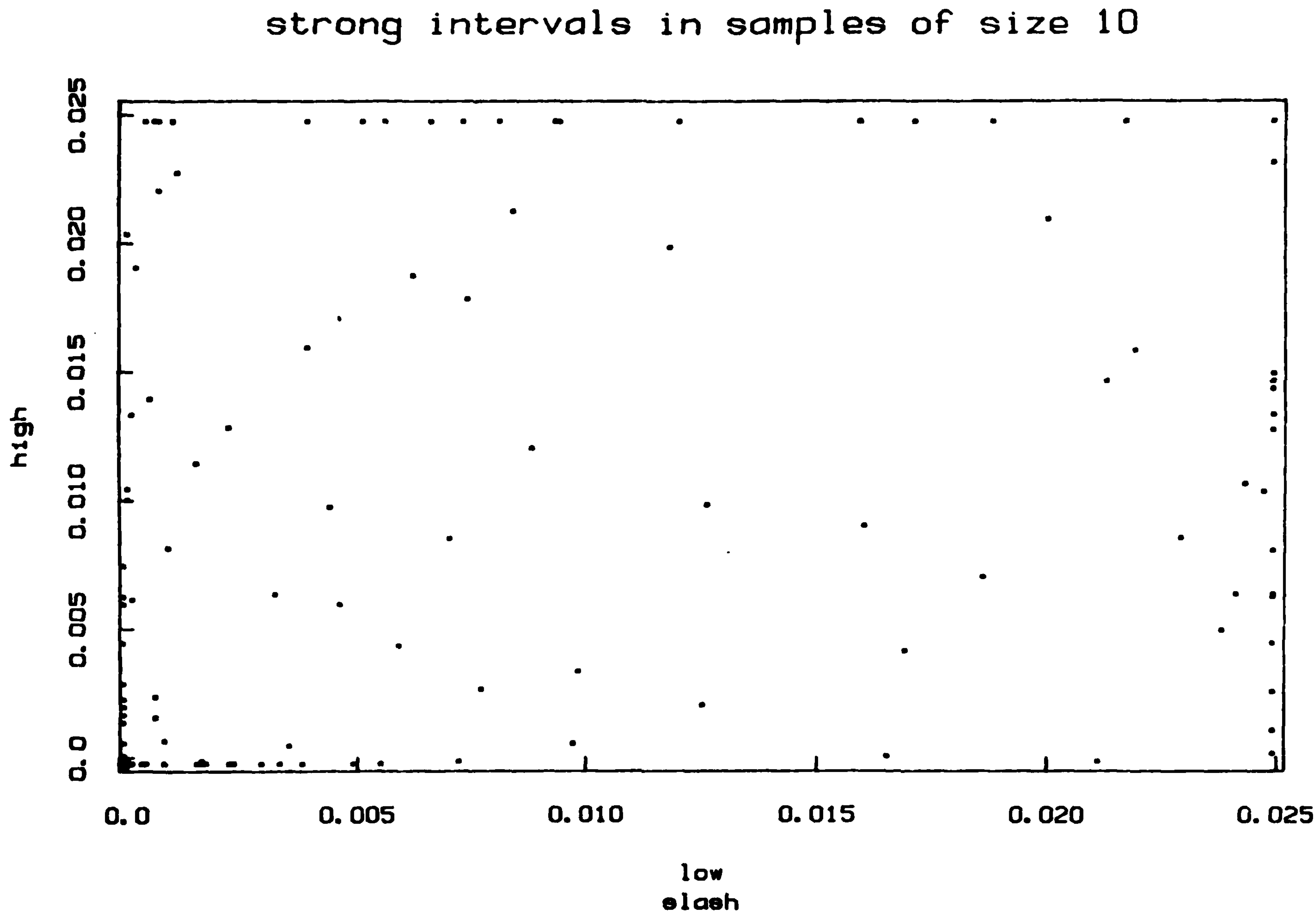
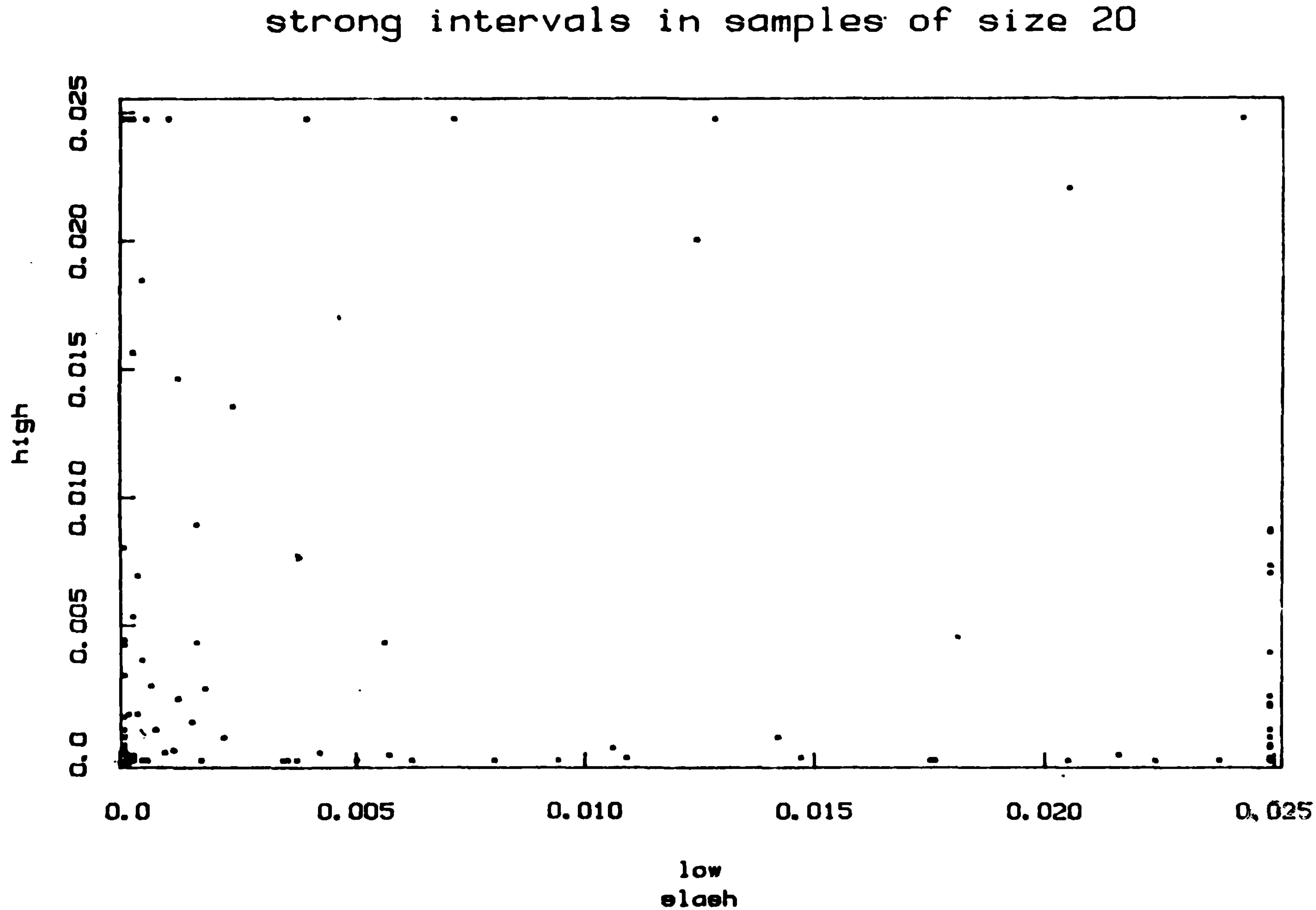


Figure 3.7: Plot of the lower vs. the upper conditional missing probabilities of the strong intervals (4.1) in the slash situation



Student's t to the strong interval is therefore usually one sided. Furthermore we notice how the increase of sample size influences the picture. More and more points get absorbed into (0.0, 0.0) -- which means they are overlong -- and the two branches we discussed are pushed toward the edges.

The above discussion has an interesting consequence: maybe it makes more sense to compromise with a less extremely heavy-tailed counterpart than the slash in samples of size 5. This finding disagrees with the fact that with bigger sample sizes the distinction between samples drawn from the slash and samples drawn from the Gaussian becomes more "obvious" -- and that compromising these two situations is therefore simpler in larger samples (see: Bell & Morgenthaler(1981)). To try this idea the "slacu"-distribution was used together with the Gaussian. This is the distribution of the ratio of a standard Gaussian and the cube root of a unit uniform -- it's density has tails like $\frac{1}{x^3}$. But still Student's t interval comes out to be very close to the strong interval -- now strong for Gaussian and slacu! Table 3.3 shows the numbers.

Table 3.3: Percentage of observed cases (a), (b) and (c)&(d) (for definition see (3.2) note, however, that the slash is replaced by the slacu)

| | Gaussian situation | | | slash situation | | |
|---|---|---|---|---|---|---|
| sample size | (a) | (b) | (c)&(d) | (a) | (b) | (c)&(d) |
| 5 | 78 $\frac{4}{5}$% | $\frac{3}{5}$% | 20 $\frac{3}{5}$% | 91 $\frac{2}{5}$% | 1 $\frac{1}{5}$% | 7 $\frac{2}{5}$% |

Surprisingly little changed by replacing the slash with the slacu. It

we want to get robustness in the sense of stable conditional coverage probability under heavy-tailed challenges, Student's t interval behaves well in samples of size 5 and leaves little scope for modifications.

### 3.2. Conditional behavior of nonparametric confidence intervals

As we pointed out in the introduction to this chapter, nonparametric confidence intervals, which are assured to have a fixed confidence level "over the sample space" for all symmetric situations, need not have a fixed coverage probability conditioned on the configuration. In this section we will see in a descriptive fashion how the intervals derived from Wilcoxon's signed rank statistic and from the sign test statistic behave conditioned on configurations.

We are interested -- as always -- in 95% confidence intervals and of course neither of the above procedures will be able to create a 95% confidence interval for samples of size 5, where the interval defined by the minimum and maximum of the sample has $1 - \frac{2}{32} = .9375$ coverage probability. In order to compare over the full range of sample sizes, we will include the "range" - interval for sample size 5. For samples of size 10 and 20 we use logistic interpolation in the Wilcoxon and binomial tables to get approximately a 95% confidence interval. On the configuration scale, i.e. expressed as intervals for $c_1 \le c_2 \le \ldots \le c_n$, we use the following intervals:

| | sign interval | Wilcoxon-interval |
|---|---|---|
| size=5 | $[c_1, c_5]$ | $[c_1, c_5]$ |
| size=10 | $[.5c_2+.5c_3, .5c_8+.5c_9]$ | $[.26w_8+.74w_9, .74w_{47}+.26w_{48}]$ |
| size=20 | $[.82c_6+.18c_7, .82c_{15}+.18c_{14}]$ | $[.34w_{54}+.66w_{53}, .34w_{157}+.66w_{158}]$ |

($w_1 \le \ldots \le w_{\frac{n(n+1)}{2}}$ denote the Walsh averages, i.e. $\frac{y_i + y_j}{2}$ ordered by value.)

It turns out that the sign interval and the Wilcoxon interval have quite different behaviors as one looks across situations and sample sizes. The following table contains the estimated variation of the conditional coverage probabilities.

Table 3.4: Hinge-spreads (see Tukey(1977)) for conditional coverage probability in %

| | | Gaussian | slash |
|---|---|---|---|
| sign intervals | size=20 | 3.50% | 3.73% |
| | size=10 | 3.66%* | 2.58%* |
| | size=5 | 2.44% | 5.35% |
| Wilcoxon intervals | size=20 | 1.25% | 3.32% |
| | size=10 | 1.30%* | 4.00%* |
| | size=5 | 2.44% | 5.35% |

(*: Entries for sample size 5 are for the "range"-interval)

The sign interval procedure is getting worse in the slash situation from samples of size 10 to 20 as far as stability of the conditional coverage level is concerned. The Wilcoxon interval seems to improve. On the whole it is surprising how little the increase in sample size stabilizes the conditional coverage levels.

Both interval procedures are bad for very small samples in the heavy-tailed slash situation. Looking across situations, the two confidence intervals are complementary as well. The sign interval is more stable in the slash for the intermediate sample size -- whereas the Wilcoxon interval is better in the Gaussian. If we adopt a measure of variation which uses more of the information in the tails of the distribution -- like the standard deviation -- the effects come out even more clearly. They can also be seen in Figures 3.8 and 3.9, which show box plots (see: Tukey(1977)) of the conditional coverage probabilities.

In these Figures we use the logistic transforms of the conditional coverage probabilities p, defined by

$$\text{logit}(p) = \log\left(\frac{p}{(1-p)}\frac{.05}{.95}\right) \tag{3.3}$$

so that, on the transformed scale, a value of zero corresponds to exactly 95% conditional coverage, whereas positive values indicate conditional coverage bigger than 95% and negative values indicate coverage smaller than 95%. We hope that the distributions will be made more symmetric by this re-expression.

In these pictures we can also see how the numbers of Table 3.4 came about. The Wilcoxon interval in samples of size 10 produces quite low conditional slash coverage probabilities in some "extreme" configurations. This is clear from the fact, that information from the smallest and largest observation are sometimes used, since they can contribute up to the $10^{th}$ and $46^{th}$ Walsh average. The median of the conditional coverage probabilities can be quite substantially bigger than 95% (the mean over configurations). This reflects the

Figure 3.8:

Logistic transforms for 150 sampled configurations of the sign interval

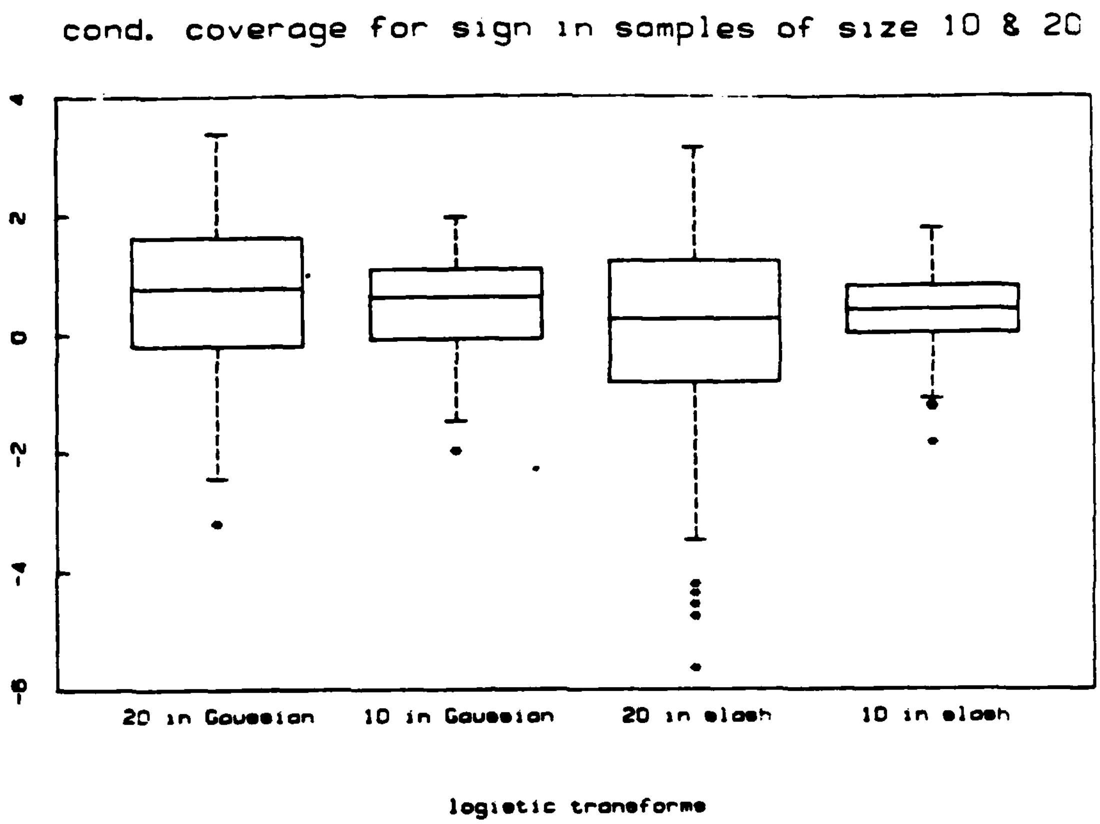
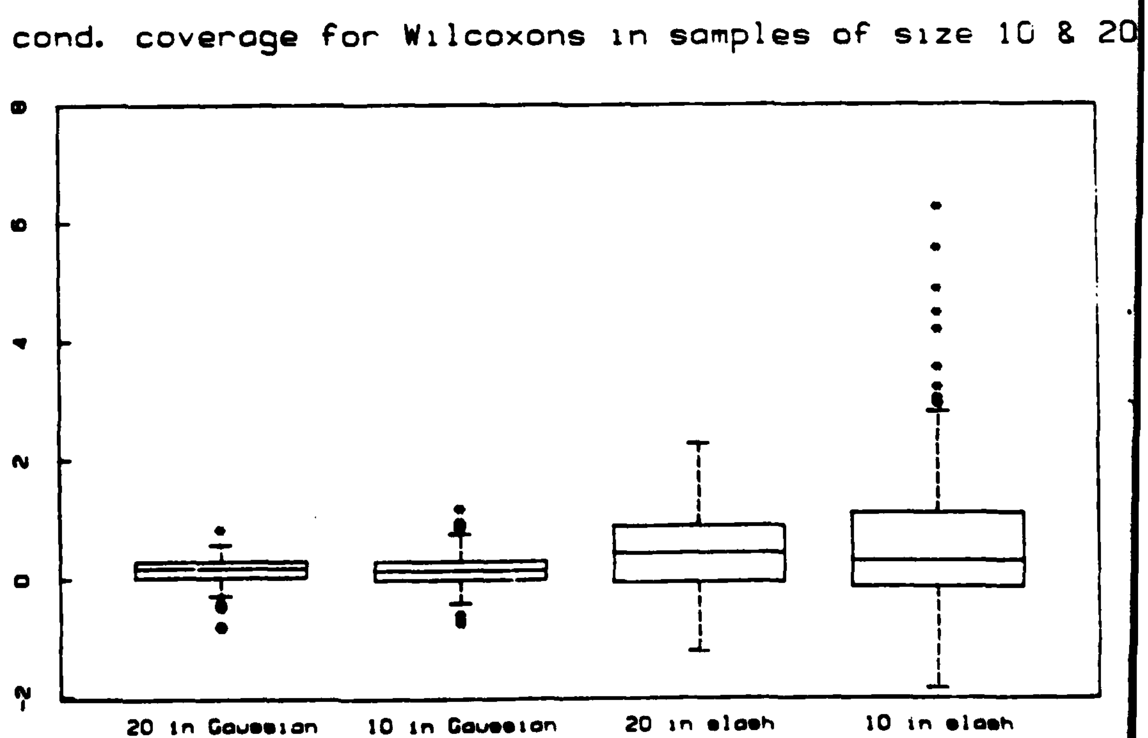



Figure 3.9:

Logistic transforms for 150 sampled configurations of the Wilcoxon interval



fact that the distribution of the conditional coverage probabilities is skewed towards low values.

Asymptotically, these differences between configurations we have observed here, disappear. If we repeatedly take large samples from a fixed shape F( ), the configurations we fall into will usually be relatively similar and the conditional coverage probability for this situation F( ) will therefore be nearly constant. All the comments we have made are hence about phenomena observed frequently only in small samples.

Figures 3.10 through 3.13 show the conditional probability of missing the true location parameter at the lower and upper tail for the two nonparametric intervals and the two situations in samples of size 10. It becomes clear now that the two situations are indeed quite different. Both procedures create rather unbalanced intervals in the slash situation -- the Wilcoxon more so than the sign. Many of the Wilcoxon intervals are too long in one direction and often too short in the other. The lines drawn in the pictures correspond to intervals with exact 95% conditional coverage probability. Most of the intervals are below this line -- and are therefore overlong. The fewer points above the line still bring the overall coverage to the nominal 95%. In Gaussian samples the Wilcoxon clearly does quite a good job and is substantially better than the sign intervals, which show a similar behavior as in the slash situation. These plots look similar in samples of size 20. In that case they are, however, a bit more concentrated for both situations.

We can summarize what we have learned up to now by saying that,

Figure 3.10: Plot of the lower vs. the upper conditional missing probabilities of the sign interval in the slash situation

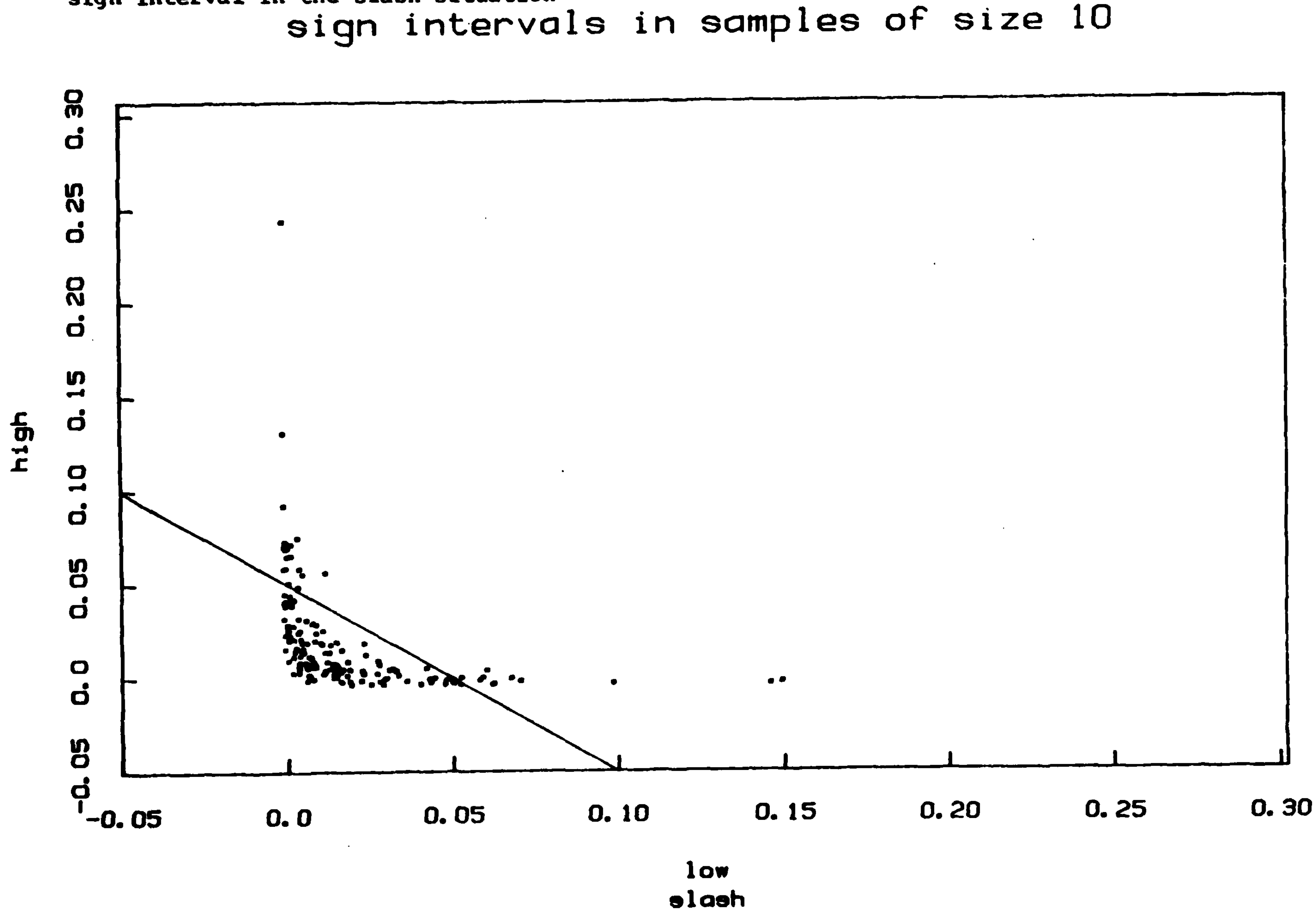


Figure 3.11: Plot of the lower vs. the upper conditional missing probabilities of the sign interval in the Gaussian situation

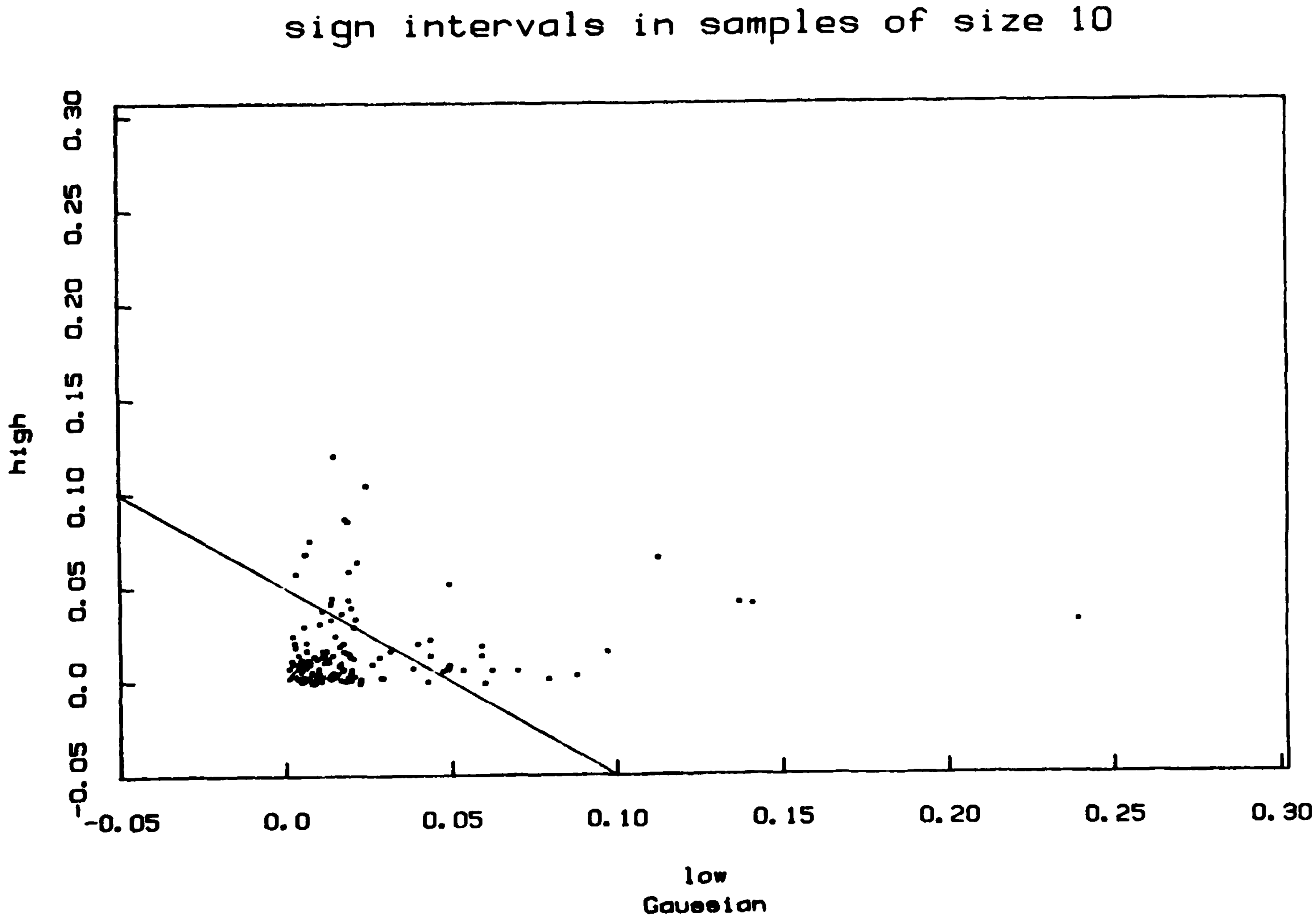


Figure 3.12: Plot of the lower vs. the upper conditional missing probabilities of the Wilcoxon interval in the slash situation

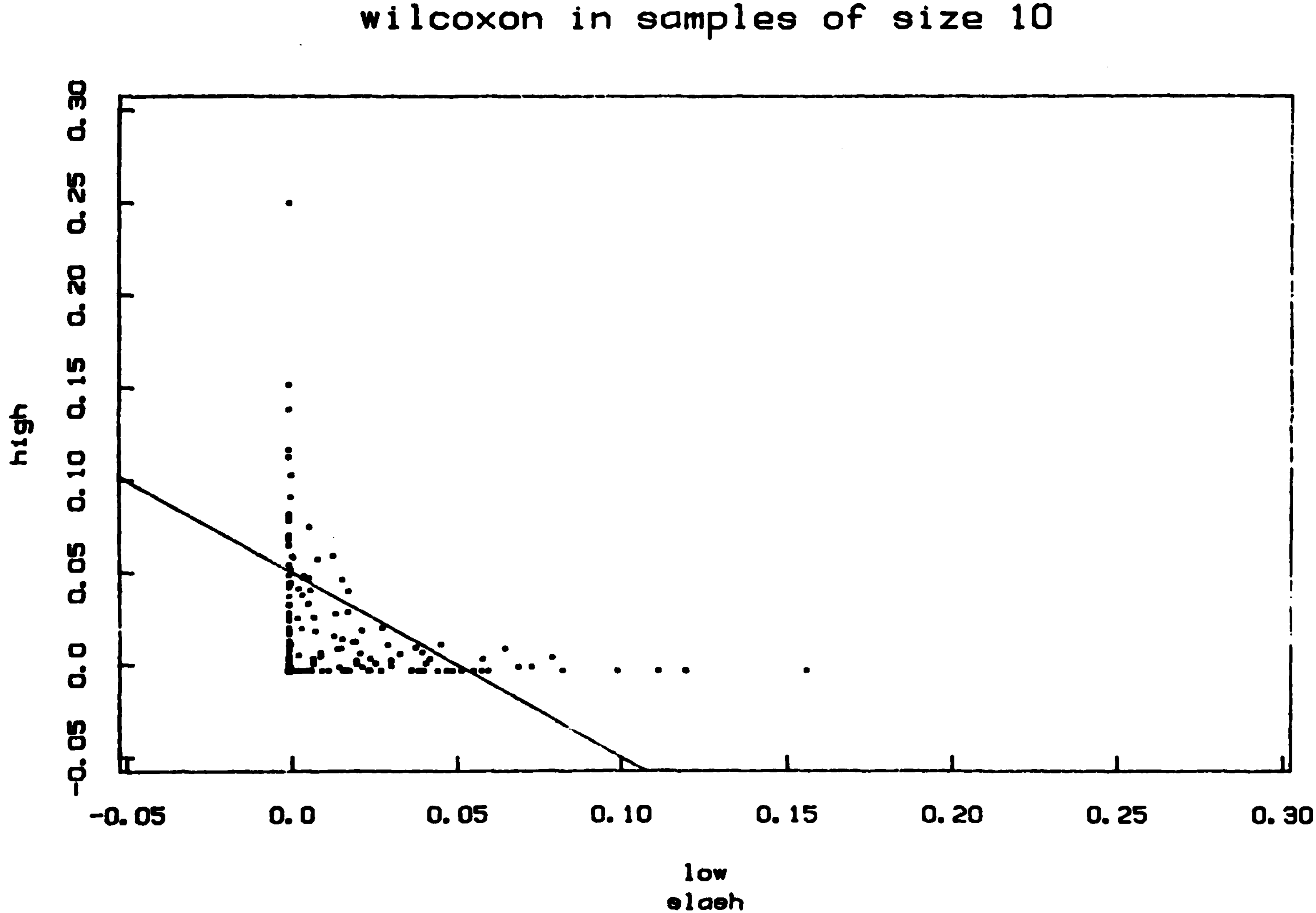
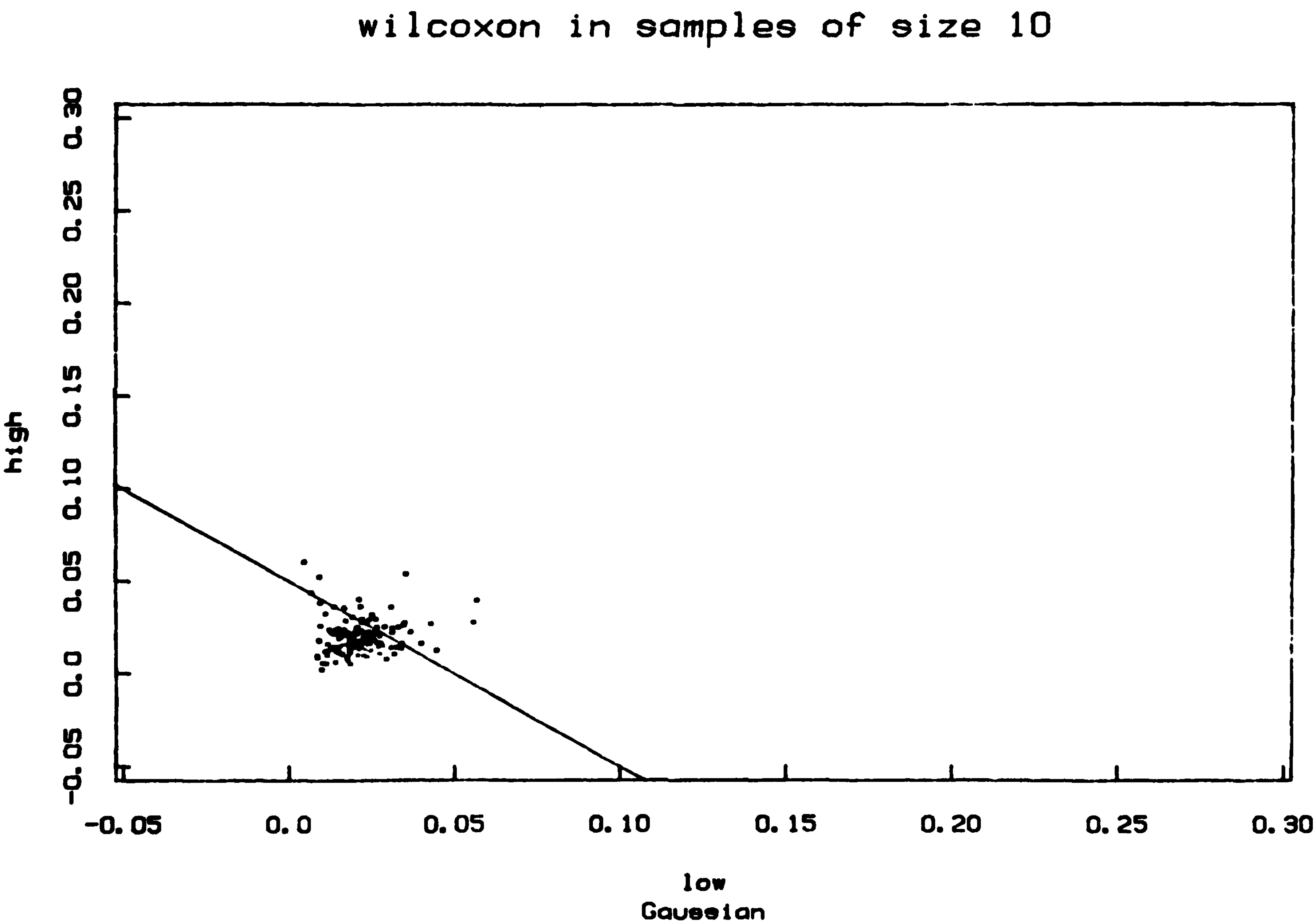


Figure 3.13: Plot of the lower vs. the upper conditional missing probabilities of the Wilcoxon interval in the Gaussian situation



even though both the sign and the Wilcoxon intervals reach a fixed confidence level across the whole sample space in all symmetric situations, there are differences as soon as we look at the conditional confidence levels. The sign intervals are more stable than the Wilcoxon intervals if we sample from the heavy-tailed slash. The opposite is true if we believe strictly in the Gaussian model. As a compromise between these two extremes, we would probably choose the Wilcoxon.

The second important thing about confidence intervals besides coverage probability is the length distribution. Table 3.5 gives the estimated mean lengths and coefficients of variation.

Table 3.5: Estimated mean length and estimated coefficient of variation

| | | Gaussian situation | | slash situation | |
|---|---|---|---|---|---|
| sign intervals | size=20 | 1.16 | 26.5% | 2.65 | 23.3% |
| | size=10 | 1.64 | 29.0% | 5.39 | 75.2% |
| Wilcoxon intervals | size=20 | .955 | 17.5% | 3.34 | 57.1% |
| | size=10 | 1.46 | 25.0% | 24.51 | 476% |

It is quite obvious that the Wilcoxon intervals are extremely long in extreme configurations of size 10. Otherwise this table confirms our view that the sign intervals are to be favored in the slash and the Wilcoxon in the Gaussian. This fact also appears in an asymptotic theory via Pitman efficiencies. The square root of these efficiencies applies to asymptotic ratios of mean lengths. The values for the Gaussian and slash are summarized in Table 3.6.

Table 3.6: Square roots of ratios of Pitman efficiencies

| methods | Gaussian | slash |
|---|---|---|
| sign vs. Wilcoxon | .8165 | 1.082 |
| sign vs. Student's t | .7975 | $\infty$ |
| Wilcoxon vs. Student's t | .9772 | $\infty$ |

We can check how well these asymptotic numbers work by comparing them to the small sample estimated ratios, as we do in Table 3.7.

Table 3.7: Estimated ratios of mean length

| | | Gaussian | slash |
|---|---|---|---|
| sign vs. Wilcoxon | size=10 | .8901 | 4.55 |
| | size=20 | .8233 | 1.2604 |
| | asympt | .8165 | 1.082 |
| sign vs. Student's t | size=10 | .8514 | 13.47* |
| | size=20 | .7966 | 53.71* |
| | asympt | .7975 | $\infty$ |
| Wilcoxon vs. Student's t | size=10 | .9566 | 15.14* |
| | size=20 | .9675 | 65.24* |
| | asympt | .9772 | $\infty$ |

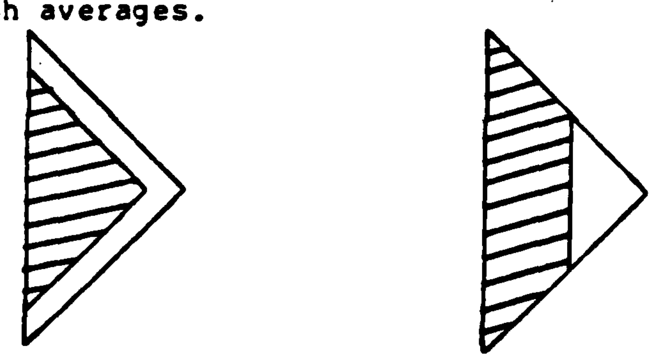
*: theoretical value is $\infty$

A comparison between the two sets of numbers shows that the asymptotic expression is a good approximation as far as samples of size 20 are concerned, but for sample size 10 the agreement is not good except in "Wilcoxon vs. Student's t".

Both nonparametric intervals we have considered up to now do not leave us entirely happy and a procedure between these two extremes

might well be a better compromise. There are at least two ways in which one can bridge the sign and Wilcoxon. One was suggested by J. W. Tukey and it works by trimming observations from the ordered sample and computing Wilcoxon's signed rank statistic on what is left over, thus omitting certain i and j entirely. The other (Policello and Hettmansperger(1976)) works by "winsorizing" the scores in Wilcoxon's signed-rank statistic. This is equivalent to omitting Walsh averages with $|i-j|>$bound. (If, for example, we put the extreme bound of 1 on the ranks, the sign statistic, i.e. the number of positive observations, comes out.)

Both of these procedures can be explained in terms of the triangle of Walsh averages.



(1) trimmed Wilcoxon (2) winsorized scores

Both only take the Walsh averages in the corresponding shaded region into account. These methods are expected to do a bit worse in the Gaussian than the Wilcoxon but to improve in the slash.

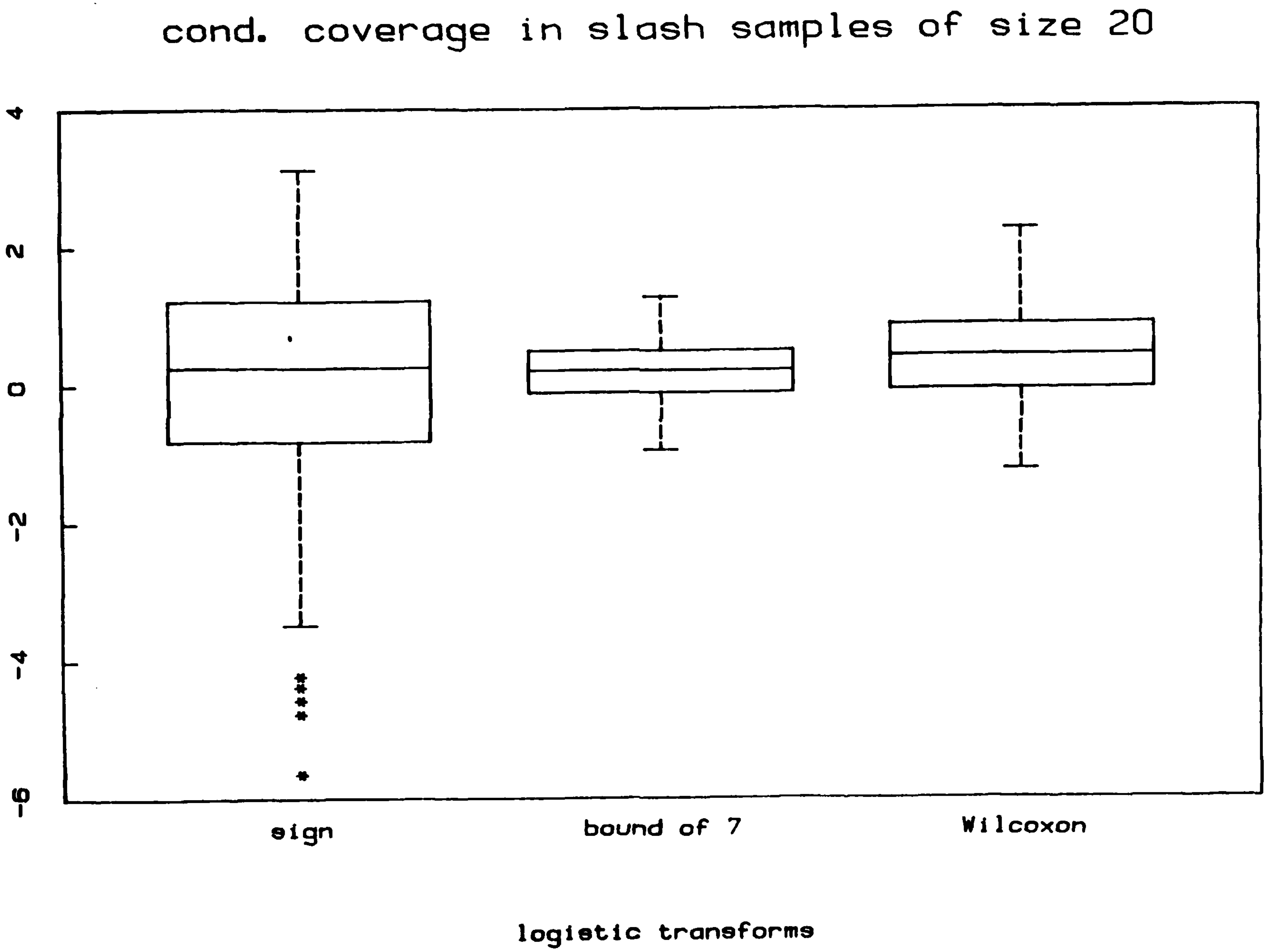
It turns out that in samples of size 10 the procedure which trims the largest and smallest observation very nearly gives the same confidence intervals as the procedure which puts a bound of 3 on the ranks. Both are worse in the Gaussian situation than the Wilcoxon, which is no surprise -- but they seem to be rather close to the sign

intervals and even a bit worse in terms of stability of the conditional coverage probabilities. In the slash situation, they are quite close to the sign intervals. In terms of expected length these "robust" intervals are an improvement (see Morgenthaler (1983)).
In samples of size 20 the two proposed modifications of the Wilcoxon test statistic produce nearly identical results. Now these "robustified" rank intervals really make an improvement. In the Gaussian situation they are between the Wilcoxon and the sign, in the slash situation they improve over both of them. Instead of having a hinge-spread of confidence levels across configurations of roughly 3% (see Table 3.4), these procedures go down to about 2.5%. Figure 3.14 shows a boxplot similar to the Figures 3.8 and 3.9.
We would therefore recommend the use of these confidence intervals which bridge the gap between the sign and the Wilcoxon procedures for larger sample sizes because in a heavy-tailed case they show an improved behavior.

## 4. Conclusions.

Strong confidence intervals are appealing from the point of view of validity of an interval estimator. Their conditional confidence coefficients given any configuration is kept above the nominal level for both situations. We only took two situations into consideration, but this is already a conclusive case. If we safeguard against heavy tailedness in the way described in this report, we do not have a difficult job. Student's t interval is already resistant as far as validity is concerned. But of course many statisticians will criticize Student's t interval as being too long in configurations

**Figure 3.14: Logistic transforms for 150 sampled configurations for three nonparametric intervals in the slash situation**



with outliers. The research described shows us the need for compromising between validity and some measure of efficiency. If we only consider validity, we automatically sacrifice efficiency in a rather extreme way. We considered three sample sizes (u = 5, 10 and 20) and it is rather striking how the problem of compromising between shapes changes with changing sample size. For samples of size 5 it turns out that Student's t intervals are nearly "optimal". As the sample size increases this no longer holds. In order to be strong, Student's t interval needs to be enlarged (usually in only one direction).

In the last section of this report we examine non-parametric confidence intervals. It becomes clear that a stable overall confidence coefficient does not ensure a good behavior of the conditional confidence coefficients.

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